LUND UNIVERSITY
Mechanics, LTH

Med lösningar

Mechanical Vibrations, FMEN11

2017-12-21, 14.00-19.00

Name (print):...
(or personal identifier)

Personal id-number:...................................
(or personal code)

Written examination with five examination tasks. Please check that all tasks are included. A clean copy of the solutions should be handed in. Write your name on all pages, number the pages and hand them in, in the correct order. Make sure that the papers are stapled together.

Summaries, equation sheets and mathematical handbooks such as TEFYMA may be used.

Results:

Task	Comment	Points (0-3)
1		
2		
3		
4		
5		
	Sum	
	Grade	

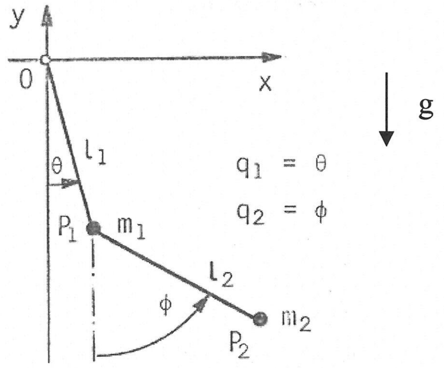
Name (signature):...   Leg:......
(Only if not anonymous)

1)

a) A double pendulum consists of two particles P_1 and P_2 with masses m_1 and m_2 respectively, connected by two strings of lengths l_1 and l_2 according to the figure below. The equations of motion for the system are:

$$a\ddot{\theta} + c\cos(\theta-\phi)\ddot{\phi} + c\sin(\theta-\phi)\dot{\phi}^2 + d\sin\theta = 0$$
$$b\ddot{\phi} + c\cos(\theta-\phi)\ddot{\theta} - c\sin(\theta-\phi)\dot{\theta}^2 + e\sin\phi = 0$$

Linearize the equations of motion at the equilibrium state $\theta = \phi = 0,\ \dot{\theta} = \dot{\phi} = 0$ and write on matrix format. Identify the mass matrix, the stiffness matrix, the damping matrix and the gyroscopic matrix.



b) Let $q = q_1,\ldots,q_n$ be generalized coordinates for a material system $\mathcal{B}$ subjected to the generalized forces $Q_i = \int_{\mathcal{B}} \boldsymbol{f}_a \cdot \dfrac{\partial \boldsymbol{r}}{\partial q_i} dm$, where $\boldsymbol{f}_a$ is the (specific) active accelerating force. The position vector of a material point $P \in \mathcal{B}$ is then given by $\boldsymbol{r} = \boldsymbol{r}(q;P)$. Let $\boldsymbol{a} = \boldsymbol{a}(q,\dot{q},\ddot{q};P)$ denote the acceleration of the material point.

Show that

$$\boldsymbol{a} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r}}{\partial q_j}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 \boldsymbol{r}}{\partial q_j \partial q_k}\dot{q}_j\dot{q}_k$$

c) During the site visit at Öresundsverket in Malmö, our hosts told us about different vibration phenomena that occur in their line of work. One example of vibrations that was discussed was axial vibrations in a turbine. Normally one would expect radial vibrations due to unbalance relative to the central shaft, but the manufacturers of the turbine had encountered problems with axial vibrations along the length of the turbine. What design change did they introduce in order to avoid these axial vibrations?

d) A simply supported beam is subjected to transverse vibrations and the following mode decomposition is used to describe the displacements $w = w(x,t)$

$$w = w(x,t) = \sum_{i=1}^{\infty} \hat{w}_i(x)\eta_i(t)$$

where $\eta_i(t)$ describes the time dependence and $\hat{w}_i(x)$ describes the mode shapes.

If the mode shapes are given by

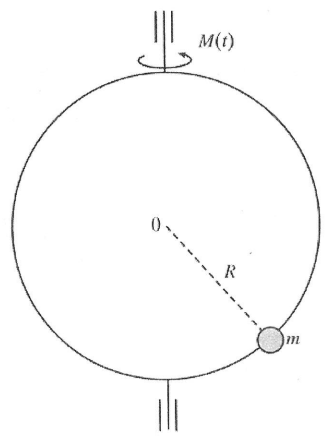
$$\hat{w}_i(x) = \sin\left(\frac{i\pi x}{L}\right), \quad i = 1, 2, \ldots$$

specify the complete expression for the orthogonality condition

$$\int_0^L \hat{w}_i(x)\hat{w}_j(x)\,dx = \ldots$$

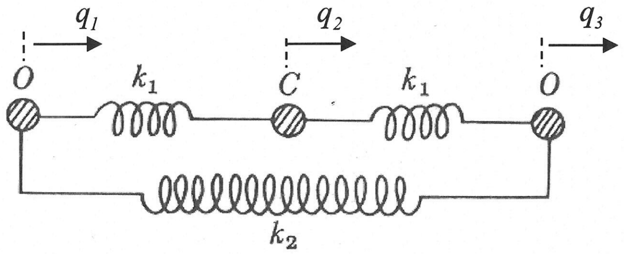
e) A mechanical system consists of a thin hoop of radius R and mass m_h that may rotate without friction around a vertical axis. A bead of mass m can slide freely (no friction) around the hoop. The hoop is acted upon by the torque $M = M(t)$ about the vertical axis. The hoop is rotating with the constant angular velocity Ω. The kinetic energy of the system may be written as

$$T = \frac{1}{4}m_h R^2\Omega^2 + \frac{1}{2}mR^2\Omega^2\sin^2\varphi + \frac{1}{2}mR^2\dot{\varphi}^2$$



Is it possible to use the potential energy V of the system $\left(\frac{\partial V}{\partial q_i} = 0\right)$ to identify equilibrium states? Motivate your answer.

f) Consider linear vibrations of the CO_2 molecule (that is, vibrations in line with the molecule, see figure below). The mass of an oxygen atom is denoted by m_O and the mass of a carbon atom by m_C. The inter-atomic forces are represented by elastic springs with spring constants k_1 and k_2, respectively, according to the figure below.



Suppose that $x_1(0) = x_2(0) = x_3(0) = 0$, and that the masses are initially at rest, except for the carbon atom which is given the initial velocity 10m/s. What is the expression for the subsequent motion $\bar{q}(t)$ of the mechanical system when contributions equal to zero are omitted from the expression? Also identify the name of the different variables in the expression.

2)

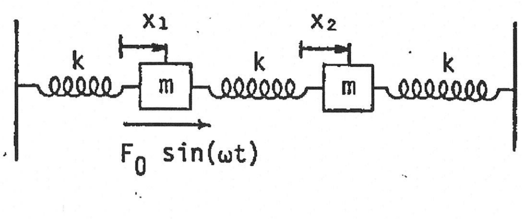
a) Consider the mechanical system in the figure below. The masses move along a fixed horizontal line and the left mass is subjected to an external harmonic force with amplitude F_0 and angular frequency ω. The mechanical system has the following mass and stiffness matrix

$$\underline{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \underline{K} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

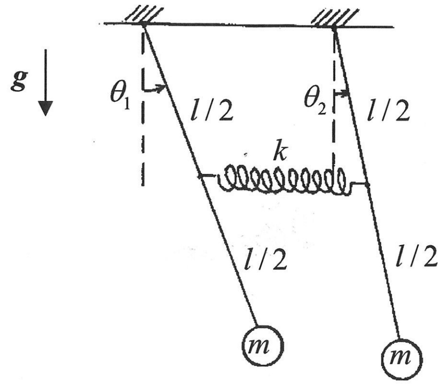
and mode shapes

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \bar{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Calculate the modal mass matrix.



b) Consider a mechanical system consisting of two identical particle pendulums with mass m and cord-length l connected with a spring whose unstressed length is equal to the distance between the points of suspension of the pendulums and with a spring constant k. Denote by θ_1 and θ_2 the angles of inclination of the pendulums, see figure below. Formulate the kinetic energy of the system.



c) An n-dimensional mechanical system, with mass-matrix $\mathbf{M}$ and frequency response function $\mathbf{F} = \mathbf{F}(\omega)$ is subjected to an external force $\mathbf{P} = \mathbf{P}_0 \sin(\omega t)$ where $\mathbf{P}_0$ is a constant amplitude vector and ω is the angular velocity. The initial conditions are

$$\mathbf{q}(0) = \mathbf{q}_0,\ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0$$

The system is "diagonizable" with natural modes:

$$\begin{array}{ll} \mathbf{x}_1, \ldots, \mathbf{x}_n & \\ s_1, \ldots, s_n & s_i = -\zeta_i \omega_{0_i} + i\omega_{d_i},\quad \omega_{d_i} = \omega_{0_i}\sqrt{1-\zeta_i^2},\quad 0 < \zeta_i < 1 \end{array}$$

The motion of the system is given by:

$$\mathbf{q} = \mathbf{q}_h + \mathbf{q}_p$$

where the homogeneous solution may be written

$$\mathbf{q}_h = \sum_{i=1}^{n} e^{-\zeta_i \omega_{0i} t}\left(a_i \cos(\omega_{di} t) + b_i \sin(\omega_{di} t)\right)\boldsymbol{x}_i$$

and with the particulate solution according to

$$\mathbf{q}_p = \mathbf{f}(t)$$

Formulate an expression for the constants a_i in the homogeneous solution.

d) An n-dimensional mechanical system with positive definite mass matrix $\underline{M}$ and positive semi-definite stiffness matrix $\underline{K}$ has the eigenfrequencies $\omega_1,\ \omega_2,\ \ldots,\omega_n$ and the modal matrix $\underline{X}=[\overline{x}_1 \ldots \overline{x}_n]$ where

$$(-\omega_i^2\underline{M}+\underline{K})\overline{x}_i=\overline{0},\quad i=1,\ldots,n$$

and

$$\underline{X}^T\underline{M}\,\underline{X}=\underline{\mu}=diag(\mu_1\ \ldots\ \mu_n)\ and\ \underline{X}^T\underline{K}\,\underline{X}=\underline{\kappa}=diag(\kappa_1\ \ldots\ \kappa_n)$$

where $\mu_1,\ \ldots,\mu_n$ and $\kappa_1,\ \ldots,\kappa_n$ are constants satisfying $\mu_i>0$ and $\kappa_i\geq 0$. The kinetic and potential energies of the system are given by

$$T=\frac{1}{2}\dot{\overline{q}}^T\underline{M}\dot{\overline{q}}\ \text{ and }\ V=\frac{1}{2}\overline{q}^T\underline{K}\overline{q}$$

respectively. Using *normal coordinates* $\overline{\eta}=(\eta_1\quad \eta_2\quad \cdot\quad \cdot\quad \cdot\quad \eta_n)^T$ defined by $\overline{q}=\underline{X}\overline{\eta}$ Show that the equation for the free motion of the system may be written

$$\ddot{\overline{\eta}}+\underline{\omega}^2\overline{\eta}=\overline{0},\quad \text{where }\ \underline{\omega}^2=\underline{\mu}^{-1}\underline{\kappa}$$

e) For vibrational motion of a bar in extension, L is the length of the bar, E is the modulus of elasticity, $A=A(x)$ is the sectional area, $k=k(x)$ is the external force density and $m=m(x)$ is the mass density. Assume that $k\equiv 0$ ("free vibrations") and consider a solution on the form ("separation of variables") $u(x,t)=\hat{u}(x)\phi(t)$, show that the following conditions are necessary, where λ is a constant.

$$-\frac{1}{m(x)}\frac{d}{dx}(EA(x)\frac{d\hat{u}(x)}{dx})=\lambda\hat{u}(x)\qquad\text{and}\qquad\frac{d^2}{dt^2}\phi(t)+\lambda\phi(t)=0$$

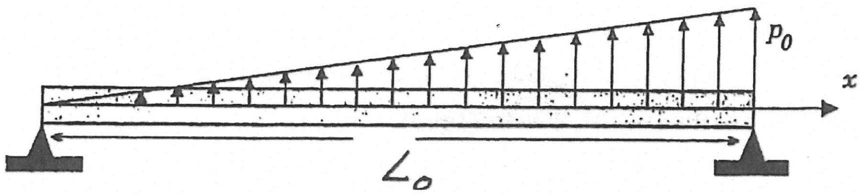
f) A mechanical system consists of ten masses $m_1,m_2,\ldots,m_{10}$ that are lined up after each other where each mass m_n is connected to the next mass $m_{(n+1)}$ using a spring with spring constant k_n, $n=1,2,\ldots,9$. Observe that mass m_1 is only connected to mass m_2 and that mass m_{10} is only connected to mass m_9. Calculations show that the system has the following natural eigenfrequencies:

$$\omega_{01}=41.1,\ \omega_{02}=79.4,\ \omega_{03}=115.8,\ \omega_{04}=157.3,\ \omega_{05}=279.4,$$
$$\omega_{06}=333.6,\ \omega_{07}=577.9,\ \omega_{08}=645.2,\ \omega_{09}=792.5,\ \omega_{010}=839.4$$

Referring to the numerical values of the eigenfrequencies, why is it reasonable to assume that the calculations are incorrect?

3) A homogeneous simply supported beam with length L_0 = 3m is subjected to an external distributed transversal force according to the figure below. The mass per unit length m_0, the modulus of elasticity E, the area moment of inertia I and the cross sectional area A_0 are known constants. Calculate the complete response to the external excitation according to the figure below.

$$p = p(x,t) = p_0 \frac{x}{L_0}$$



From the homogeneous solution it is known that the mode shapes may be written as

$$\hat{w}_k(x) = \sin\left(\frac{k\pi x}{L_0}\right)$$

and

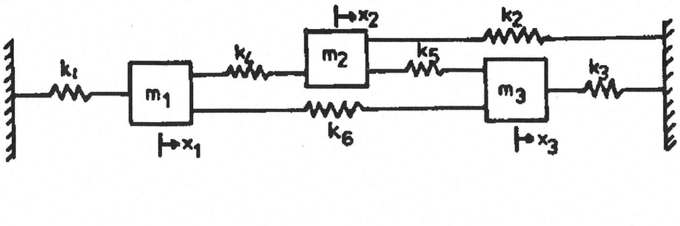
$$\frac{d^4\hat{w}}{dx^4} = \mu_k^4 \quad \text{where} \quad \mu_k = \frac{k\pi}{L_0} \quad \text{and} \quad EI\mu_k^4 = \omega_k^2 m_0$$

Assume that the beam has an initial velocity when the load is applied so that

$$w(x,0) = 0,\ \dot{w}(x,0) = \sum_{k=1}^{\infty} \sin(\frac{k\pi x}{L_0})10 \text{ m/s}$$

4) Consider the following model of a mechanical system consisting of three masses connected with springs according to the figure below.

$$m_1 = 0.5\text{kg},\ m_2 = 1.0\text{kg},\ m_3 = 1.5\text{kg}$$
$$k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 1000\text{N/m} \qquad (1)$$



Introduce damping to the system by adding a viscous damper with damping coefficient c in between masses m_2 and m_3, in parallel with spring k_5.

a) Use coordinates x_1, x_2, x_3 to formulate the equations of motion for the system. (Introduction of mass matrices etcetera should be motivated using Lagrange´s equations, expressions of kinetic- and potential energy or free-body diagrams with accompanying equations of motion.)
b) Describe how to calculate the natural frequencies and their corresponding mode shapes, the damped natural frequencies and their corresponding mode shapes and the modal relative dampings.
c) Describe how to calculate the component $F_{22} = F_{22}(\omega)$ of the frequency response matrix.

In your description, please include the equations that have to be solved and present a list of names of all variables that are introduced in your equations. Necessary relationships that have to be used to solve the problem should be presented. Matlab and similar programs are just tools. Your task is to present a systematic description for the reader how to solve b) and c) using the results found in a).

5) In order to measure forces exerted by living nerve cells on their environment, nano beams are used where the deflections of the beams are measured. In order to calculate the force, physical properties of the nano beams need to be recorded. A first step is to formulate a mathematical model of the vibration of the nano beams. Formulate an expression for the mode shapes of transverse (bending) vibrations of a nano beam which is clamped (fixed) at one end and free at the other end. Also formulate an expression from which it is possible to numerically calculate the natural frequencies for the nano beams. The expressions may include mathematical functions such as for example sine and cosine, as well as the length of the nano wire: L, the sectional area: A, the sectional area moment-of-inertia: I, the material density: ρ and the modulus of elasticity: E. Assume that L, A, I, ρ, and E may be considered as constants. With given numerical values for the constants it should be possible to plot the theoretical mode shapes using your expression.

ρ, E, A, L, I

Mekaniska Vibrationer 2017

1)

a)

$$\underbrace{\begin{bmatrix} a & c \\ c & b \end{bmatrix}}_{\underline{M}} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \underbrace{\begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix}}_{\underline{K}} \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{C} = \underline{G} = \underline{0}$$

b)

$$\dot{\bar{r}} = \underbrace{\frac{\partial \bar{r}}{\partial t}}_{=0} + \sum_{i=1}^{n} \frac{\partial \bar{r}}{\partial q_i} \dot{q}_i$$

$$\ddot{\bar{r}} = \sum_{i=1}^{n} \frac{\partial \bar{r}}{\partial q_i} \ddot{q}_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \bar{r}}{\partial q_i \partial q_j} \dot{q}_i \dot{q}_j$$

c) Every second turbine blade was slightly altered so that vibrations could not spread.

d)

$$\int_0^L \hat{w}_i \hat{w}_j \, dx = \begin{cases} 0 & i \neq j \\ \frac{L}{2} & i = j \end{cases}$$

e) No, To parts present makes V unsuitable

f)

$$\bar{q}_h = \left(\sum_{i=1}^{m} \frac{\bar{X}_i \bar{X}_i^T \underline{M}}{\mu_i} t + \sum_{i=m+1}^{n} \frac{\bar{X}_i \bar{X}_i^T \underline{M}}{\mu_i \omega_i} \sin \omega_i t \right) \dot{\bar{q}}_0$$

$\bar{X}_i$: Mode shapes

$\underline{M}$: Mass matrix

ω_i : Natural undamped frequencies

t : Time

$\dot{\bar{q}}_0$: Initial velocity vector

μ_i : Modal mass ($\mu_i = \bar{X}_i^T \underline{M} \bar{X}_i$)

2)

a)

$$\mu_i = \bar{X}_i^T \underline{M} \bar{X}_i \Rightarrow \underline{\mu} = \begin{pmatrix} 2m & 0 \\ 0 & 2m \end{pmatrix}$$

$$[1\ 1] \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [m\ m] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2m$$

b)

$$T = \frac{1}{2} m \bar{V}_1 \cdot \bar{V}_1 + \frac{1}{2} m \bar{V}_2 \cdot \bar{V}_2$$

$$\bar{V}_1 = l\dot{\theta}_1 \cos\theta_1 \bar{e}_x + l\dot{\theta}_1 \sin\theta_1 \bar{e}_y$$

$$\bar{V}_2 = l\dot{\theta}_2 \cos\theta_2 \bar{e}_x + l\dot{\theta}_2 \sin\theta_2 \bar{e}_y$$

$$\bar{V}_1 \cdot \bar{V}_1 = (l\dot{\theta}_1)^2 \cos^2\theta_1 + (l\dot{\theta}_1)^2 \sin^2\theta_1 = l^2 \dot{\theta}_1^2$$

$$\bar{V}_2 \cdot \bar{V}_2 = (l\dot{\theta}_2)^2 \cos^2\theta_2 + (l\dot{\theta}_2)^2 \sin^2\theta_2 = l^2 \dot{\theta}_2^2$$

$$T = \frac{m}{2} l^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)$$

c)

$$\bar{q}_0 = \sum_{i=1}^{n} a_i \bar{X}_i + \bar{f}(\theta)$$

$$\bar{X}_j^T \underline{M} \bar{q}_0 = \sum_{i=1}^{n} a_i \bar{X}_j^T \underline{M} \bar{X}_i + \bar{X}_j^T \underline{M} \bar{f}(\theta) = a_j \mu_j + \bar{X}_j^T \underline{M} \bar{f}(\theta)$$

$$\Rightarrow a_j = \frac{\bar{X}_j^T \underline{M} \bar{q}_0 - \bar{X}_j^T \underline{M} \bar{f}(\theta)}{\mu_j}$$

d)

$$\bar{q} = \underline{X} \bar{\eta} \Rightarrow \ddot{\bar{q}} = \underline{X} \ddot{\bar{\eta}} \Rightarrow \underline{M} \underline{X} \ddot{\bar{\eta}} + \underline{K} \underline{X} \bar{\eta} = \bar{0}$$

$$\Rightarrow \underline{X}^T \underline{M} \underline{X} \ddot{\bar{\eta}} + \underline{X}^T \underline{K} \underline{X} \bar{\eta} = \bar{0} \Rightarrow \underline{\mu} \ddot{\bar{\eta}} + \underline{K} \bar{\eta} = \bar{0}$$

$$\Rightarrow \ddot{\bar{\eta}} + \underbrace{\underline{\mu}^{-1} \underline{K}}_{\omega^2} \bar{\eta} = \bar{0}$$

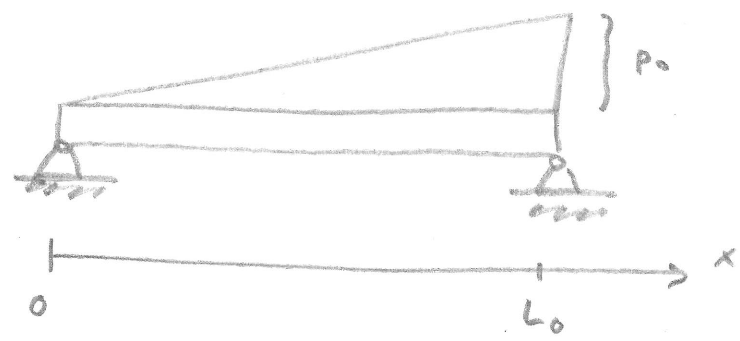
e)

$$\frac{\partial}{\partial x}\left(EA \frac{\partial(\hat{u}\phi)}{\partial x}\right) = \frac{\partial^2(\hat{u}\phi)m}{\partial t^2} \Rightarrow \frac{\frac{EA}{m} \frac{\partial^2 \hat{u}}{\partial x^2}}{\hat{u}} = \frac{\frac{\partial^2 \phi}{\partial t^2}}{\phi} = -\lambda$$

$$\Rightarrow -\frac{EA}{m} \frac{d^2 \hat{u}}{dx^2} = \lambda \hat{u}, \quad \frac{d^2 \phi}{dt^2} + \lambda \phi = 0$$

f) No $\omega_{0i} = 0$, which would be expected (rigid body mode)

3)



Given information:

E: elastic modulus
I: Area moment of inertia
L_0: Length
m_0: Mass per unit length
A_0: Cross sectional area

$\underbrace{\qquad\qquad}_{\text{Known constants}}$

$$p(x,t) = p_0 \frac{x}{L_0}$$

- From the homogeneous solution:

$$\hat{w}_k = \sin\left(\frac{k\pi x}{L_0}\right), \quad \frac{d^4 \hat{w}_k}{dx^4} = \mu_k^4 \hat{w}_k, \quad \mu_k = \frac{k\pi}{L_0}, \quad EI\mu_k^4 = \omega_k^2 m_0$$

- Initial conditions:

$$w(x,0) = 0, \quad \dot{w}(x,0) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L_0}\right) 10 \text{ m/s}$$

- Calculate the full response
- Governing equation:

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) + m_0 \ddot{w}(x,t) = p + \underbrace{\frac{\partial h}{\partial x}}_{=0 \text{ here}}$$

- Separable solution:

$$w(x,t) = \sum_{k=1}^{\infty} \hat{w}_k(x)\, \eta_k(t)$$

- Insert and find:

$$EI \sum_{k=1}^{\infty} \underbrace{\frac{\partial^4 \hat{w}_k}{\partial x^4}}_{\mu_k^4 \hat{w}_k} \eta_k + m_0 \sum_{k=1}^{\infty} \hat{w}_k \ddot{\eta}_k = p$$

0,5

3)

- Also use $\mu k^4 = \dfrac{\omega_k^2 m_0}{EI}$

$$\Rightarrow \sum_{k=1}^{\infty}\left(\omega_k^2 \hat{w}_k \eta_k + \hat{w}_k \ddot{\eta}_k\right) = \frac{p}{m_0} \qquad 0{,}5$$

- Use orthogonality condition:

$$\sum_{k=1}^{\infty}\int_0^{L_0}\left(\omega_k^2 \eta_k + \ddot{\eta}_k\right)\hat{w}_k \hat{w}_j \, dx = \int_0^{L_0} \frac{p_0 x}{m_0 L_0}\hat{w}_j \, dx$$

$$\Rightarrow \left(\omega_j^2 \eta_j + \ddot{\eta}_j\right)\frac{L_0}{2} = \frac{p_0}{m_0 L_0}\int_0^{L_0} x \cdot \sin\left[\frac{j\pi x}{L_0}\right] dx \qquad 0{,}5$$

- See page 952 in "Mathematical handbook for Scientists and Engineers":

$$\Rightarrow \ddot{\eta}_j + \omega_j^2 \eta_j = \frac{2p_0}{m_0 L_0^2}\left[\frac{\sin\left(\frac{j\pi x}{L_0}\right)}{\left(\frac{j\pi}{L_0}\right)^2} - \frac{x\cos\left(\frac{j\pi x}{L_0}\right)}{\left(\frac{j\pi}{L_0}\right)}\right]_0^{L_0} = \qquad 0{,}5$$

$$= \frac{2p_0}{m_0 L_0^2}\left[\frac{0 - L_0\cos\left[\frac{j\pi L_0}{L_0}\right]}{\frac{j\pi}{L_0}}\right] = \underbrace{\frac{2p_0}{m_0 j\pi}\left(-1\right)^{j+1}}_{\text{Independent on time} = \mathbb{C}}, \; j = 1, 2, \ldots$$

- Initial conditions:

$$\eta_j(0) = 0 \, , \; \dot{\eta}_j(0) = 10$$

3]

- Ansatz: $\eta_j = \mathbb{A} \sin \omega_j t + \mathbb{B} \cos \omega_j t + \frac{\mathbb{C}}{\omega_j^2}$

$$\underbrace{\mathbb{A} \sin \omega_j t + \mathbb{B} \cos \omega_j t}_{\eta_h} + \underbrace{\frac{\mathbb{C}}{\omega_j^2}}_{\eta_p}$$

- Insert initial conditions:

$$\eta_j(0) = \mathbb{B} + \frac{\mathbb{C}}{\omega_j^2} = 0 \Rightarrow \mathbb{B} = -\frac{\mathbb{C}}{\omega_j^2}$$

$$\dot{\eta}_j(0) = \omega_j \mathbb{A} = 10 \Rightarrow \mathbb{A} = \frac{10}{\omega_j}$$

0,5

$$\Rightarrow \eta_j(t) = \frac{10}{\omega_j} \sin \omega_j t - \frac{\mathbb{C}}{\omega_j^2} \cos \omega_j t + \frac{\mathbb{C}}{\omega_j^2} =$$

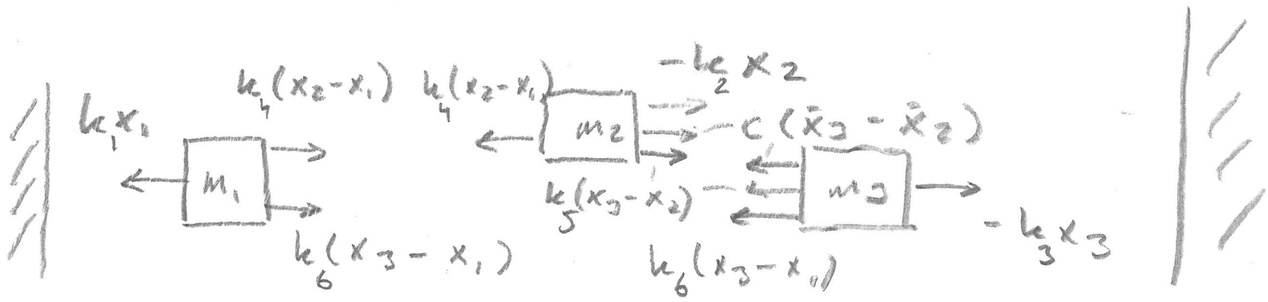
$$= \frac{2 p_0}{m_0 j \pi \omega_j^2} (-1)^{j+1} \left(1 - \cos \omega_j t\right) + \frac{10}{\omega_j} \sin \omega_j t$$

$$\Rightarrow w(x,t) = \sum_{k=1}^{\infty} \sin\left[\frac{k\pi x}{L_0}\right] \left[\frac{2 p_0}{m_0 j \pi \omega_j^2} (-1)^{j+1} \left(1 - \cos \omega_j t\right) + \frac{10}{\omega_j} \sin \omega_j t\right]$$

0,5

4)

a)



$$m_1\ddot{x}_1 = k_4(x_2-x_1)+k_6(x_3-x_1)-k_1x_1$$

0,5
$$m_2\ddot{x}_2 = -k_4(x_2-x_1)-k_2x_2+k_5(x_3-x_2)+c(\dot{x}_3-\dot{x}_2)$$
$$m_3\ddot{x}_3 = -k_3x_3-k_6(x_3-x_1)-k_5(x_3-x_2)-c(\dot{x}_3-\dot{x}_2)$$

$$\bar{0} = \underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{\underline{M}} \underbrace{\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix}}_{\ddot{\bar{X}}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & c & -c \\ 0 & -c & c \end{bmatrix}}_{\underline{C}} \underbrace{\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix}}_{\dot{\bar{X}}} + \underbrace{\begin{bmatrix} (k_1+k_4+k_6) & -k_4 & -k_6 \\ -k_4 & (k_2+k_4+k_5) & -k_5 \\ -k_6 & -k_5 & (k_3+k_5+k_6) \end{bmatrix}}_{\underline{K}} \underbrace{\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}}_{\bar{X}}$$

b) Natural frequencies: ω_{0i}

Undamped mode shapes: $\bar{X}_{0i}$

Damped natural frequencies: ω_{di}

Damped mode shapes: $\bar{X}_{di}$

Modal relative dampings: ξ

0,5 Solve $\det(-\omega_{0i}^2\underline{M}+\underline{K})=0 \Rightarrow \omega_{0i}$, insert ω_{0i} in
$(-\omega_{0i}\underline{M}+\underline{K})\bar{X}_{0i}=\bar{0} \Rightarrow \bar{X}_{0i}$

0,5

Solve $\det(-s_i^2\underline{M}+s_i\underline{C}+\underline{K})=0 \Rightarrow s_i=\sigma_i+i\omega_{di}$

Identify

0,5 $\xi_i = \dfrac{1}{\sqrt{1+\left(\dfrac{\omega_{di}}{\sigma_i}\right)^2}}$

Insert s_i into $(-s_i^2\underline{M}+s_i\underline{C}+\underline{K})\bar{X}_{di}=\bar{0}$
$\Rightarrow \bar{X}_{di}$

0,5

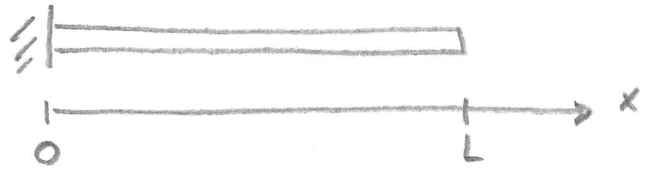
c) Find:
$$\begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{bmatrix}$$

$$\underline{F}(\omega)=\underline{A}(i\omega)=\sum_{i=1}^{n}\frac{\bar{X}_{di}\bar{X}_{di}^T}{(i\omega-s_i)\alpha_i}+\frac{\bar{X}_{di}^*\bar{X}_{di}^{*T}}{(i\omega-s_i^*)\alpha_i^*}$$

0,5

$\alpha_i = \bar{X}_{di}^T\underline{C}\bar{X}_{di}+2s_i\bar{X}_{di}^T\underline{M}\bar{X}_{di}$ where $*$ is complex conjugate

5)



- Governing equation:

$$\frac{\partial^2}{\partial x^2}\left(EI\,\frac{\partial^2 w}{\partial x^2}\right) + \ddot{w}\,m = \underbrace{p + \frac{\partial h}{\partial x}}_{=0 \text{ here}}$$

- Assume: $w(x,t) = \hat{w}(x)\sin\omega t$

$$\Rightarrow \quad EI\,\frac{\partial^4 \hat{w}}{\partial x^4}\sin\omega t - m\omega^2\,\hat{w}\sin\omega t = 0$$

0,5

$$\Rightarrow \quad \frac{\partial^4 \hat{w}}{\partial x^4} = \frac{m\omega^2}{EI}\,\hat{w} = \mu^4\,\hat{w}$$

- General solution:

(*) $$\hat{w}(x) = A\sin\mu x + B\cos\mu x + C\sinh\mu x + D\cosh\mu x$$

0,5

- Boundary conditions:

1) $w(0,t) = 0 \quad \Rightarrow \quad \hat{w}(0) = 0$

1,5

2) $\dfrac{\partial w(0,t)}{\partial x} = 0 \quad \Rightarrow \quad \dfrac{\partial \hat{w}}{\partial x}(0) = 0$

3) $M(L,t) = 0 \quad \Rightarrow \quad -EI\,\dfrac{\partial^2 w}{\partial x^2}(L,t) = 0 \quad \Rightarrow \quad \dfrac{\partial^2 \hat{w}}{\partial x^2}(L) = 0$

4) $S(L,t) = 0 \quad \Rightarrow \quad \dfrac{\partial}{\partial x}\left(-EI\,\dfrac{\partial^2 w}{\partial x^2}(L,t)\right) = 0 \Rightarrow \dfrac{\partial^3 \hat{w}(L)}{\partial x^3} = 0$

- Insert:

1) $\hat{w}(0) = B + D = 0 \quad \Rightarrow \quad D = -B$

2) $\dfrac{\partial \hat{w}(0)}{\partial x} = \mu(A + C) = 0 \quad \Rightarrow \quad C = -A$

5

3) $$\frac{\partial^2 \hat{w}(L)}{\partial x^2} = \mu^2 \left[-A\sin\mu L - B\cos\mu L + \underbrace{C}_{-A}\sinh\mu L + \underbrace{D}_{-B}\cosh\mu L \right] = 0$$

4) $$\frac{\partial^3 \hat{w}(L)}{\partial x^3} = \mu^3 \left[-A\cos\mu L + B\sin\mu L + \underbrace{C}_{-A}\cosh\mu L + \underbrace{D}_{-B}\sinh\mu L \right] = 0$$

3) $\Rightarrow$ $$-A(\sin\mu L + \sinh\mu L) - B(\cos\mu L + \cosh\mu L) = 0$$

$\Rightarrow$ $$B = -A\frac{(\sin\mu L + \sinh\mu L)}{(\cos\mu L + \cosh\mu L)} \quad \boxed{*}$$

4) $\Rightarrow$ $$-A(\cos\mu L + \cosh\mu L) - B(-\sin\mu L + \sinh\mu L) = 0$$

insert $\boxed{*}$ $\Rightarrow$ $$-A\underbrace{\left\{ (\cos\mu L + \cosh\mu L) - \frac{(\sin\mu L + \sinh\mu L)(-\sin\mu L + \sinh\mu L)}{(\cos\mu L + \cosh\mu L)} \right\}}_{\text{Solvable numerically} \Rightarrow \mu L} = 0$$

0,5

insert into $\omega = \sqrt{\frac{\mu^4 EI}{m}} \rightarrow \omega$

$\hat{w}(x) =$ $\triangle$ with constants as above.