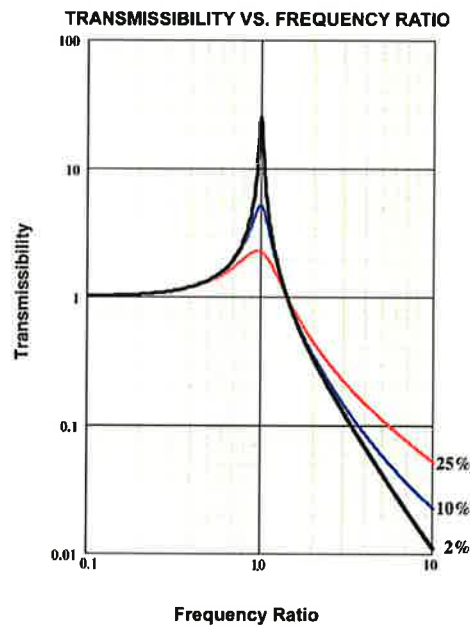


LECTURE NOTES ON MECHANICAL VIBRATIONS



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Autumn 2014

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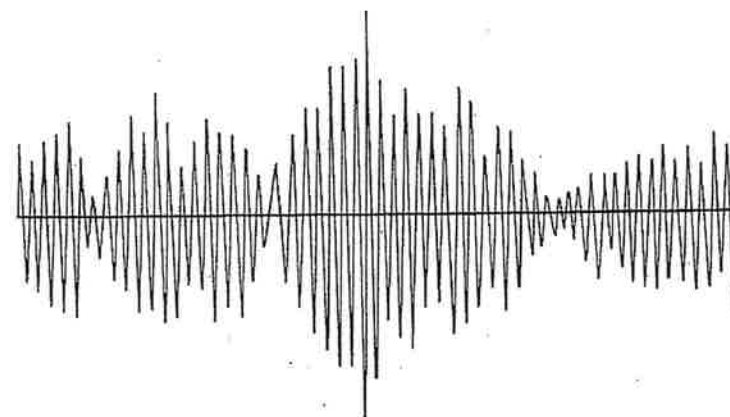
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INTRODUCTION

INTRODUCTION

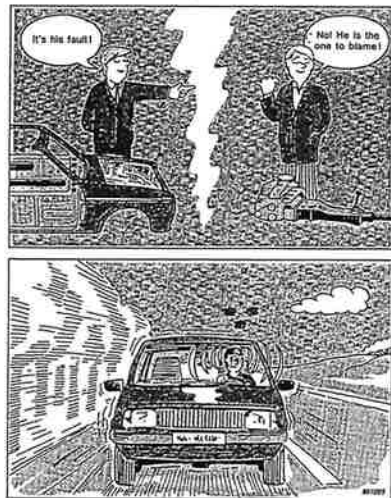
Mechanical vibrations

Due to the increase in governmental regulations and in market competition and demands, the engineers of today are faced with far more complex designs than ever before. In the design of machines, for instance in the automotive industry, the issues of comfort and service life are important. Both are intimately connected to mechanical vibrations. Sources to the vibrations are often related to engine characteristics, to aerodynamics and, in the case of a road vehicle, to the tire-road interaction. An often profitable strategy for the reduction of noise and

vibrations will, of course, be to address the sources themselves and try to reduce their activity. This is however only possible to a certain degree and to obtain a good enough result one has to look at the entire machine from a dynamical point of view. We need to know the vibration characteristics of the machine, such as the structural resonances. These resonances may lead to amplifications of normal service loads with unacceptable mechanical accelerations, stresses or sound-levels as a result.

Who is responsible for the problem?

(From Brüel & Kjaer [4]).



The theory of vibration dates back to 1877 when Lord Rayleigh published his treatise Theory of Sound. where he formulated the fundamental concepts and principles of oscillations in discrete and continuous mechanical systems. In the 1930's, when engineers were struggling to minimize the weight of aircrafts, the engineering significance of the subject of vibration and structural dynamics became more evident, but applications were highly constrained: due to lack of computational means, thus restricted to analytical solutions or only a small number of degrees of freedom discretization using the Rayleigh-Ritz techniques.

Experimental modal analysis was developed in the 1940's but was at

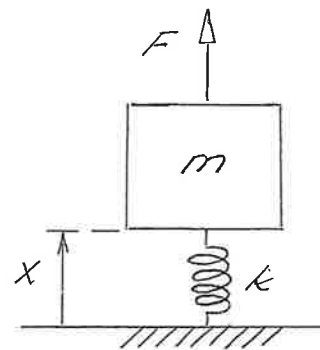
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this time performed with very primitive equipment and it was very time-consuming. In the 1960's the appearance of computing hardware replaced the traditional methods with so-called matrix methods arising from the discretizations of the mechanical systems. Especially the Finite Element (FE) method for analyzing structures made it possible to deal with problems of increasing size. Also the experimental modal analysis entered the modern era with mathematical tools like the Fast Fourier Transform (FFT), see the references Geradin & Rixen [1], Ewins [2] and Allemang [3]. Today, with the evolution of computation hardware and matrix computation procedures, it is routine to pre-

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dict the dynamic behaviour of structures using computers and this has replaced many of the costly experimental modal analyses. However, to be able to perform these types of analyses with good quality, a thorough understanding of the theory of vibrations is required. It is the purpose of this course to lay the groundwork for this.

In the basic course in Mechanics we have learned about the simple one-dimensional system. See Meriam & Kraige [5], chap. 8.



A body with the mass m is connected to a foundation with a linear elastic spring

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with spring constant k . We assume that the body is subjected to the external force F , see figure. The equation of motion for this system may be written:

$$m\ddot{x} + kx = F \quad (I.1)$$

By taking the external force equal to zero we look at the free vibrations of the body given by the solution of the differential equation (I.1) with $F=0$, i.e. by the motion: $(x(0)=x_0, \dot{x}(0)=v_0)$

$$x(t) = x_0 \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t \quad (I.2)$$

where

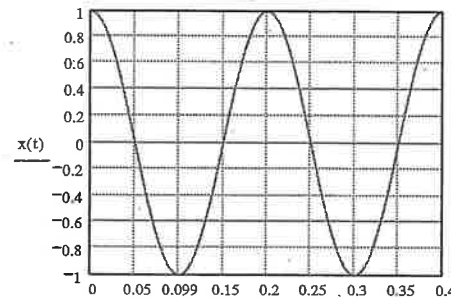
$$\omega_0 = \sqrt{\frac{k}{m}} \quad (I.3)$$

is the natural frequency ('eigenfrequency') of the system and the con-

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stants x_0, v_0 are given by the initial conditions (at time $t=0$):

$$x(0) = x_0, \quad \dot{x}(0) = v_0 \quad (I.4)$$



$$m = 20 \text{ kg}$$

$$k = 20 \text{ kNm}^{-1}$$

$$\omega_0 = 31.65 \text{ s}^{-1}$$

$$x_0 = 1$$

$$v_0 = 0$$

If we apply an external force:

$$F(t) = \hat{F} \sin \omega t \quad (I.5)$$

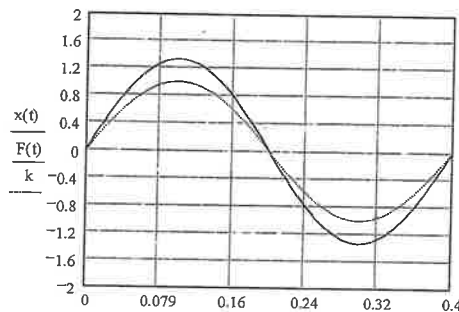
where $\hat{F}$ is the amplitude of the force and ω is the angular frequency of the force we get the following vibration (the 'steady-state' vibration)

$$x(t) = g\left(\frac{\omega}{\omega_0}\right) \frac{F(t)}{k} = g\left(\frac{\omega}{\omega_0}\right) \frac{\hat{F}}{k} \sin \omega t \quad (I.6)$$

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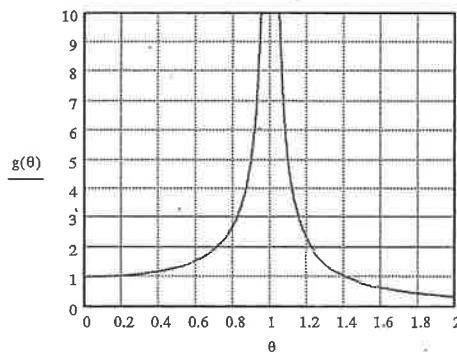
where $g = g(\theta)$ is the so-called magnification factor (or amplitude ratio)

$$g(\theta) = \frac{1}{1 - \theta^2} \quad (I.7)$$



$$\omega = 0.5\omega_0$$

$$g = 1.333$$



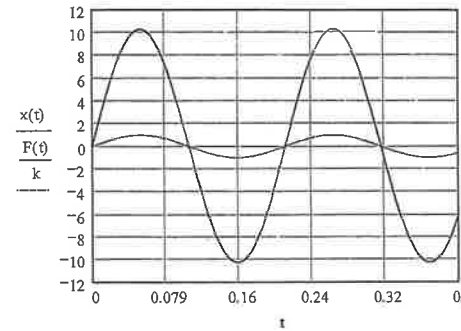
The magnification factor

$$g(0.5) = 1.333$$

If we take ω close to ω_0 then the magnification factor will be

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large since formally: $g(\theta) \rightarrow \infty$ as $\theta \rightarrow 1$.



$$\omega = 0.95\omega_0$$

$$g \approx 10.256$$

The amplitude of the response $x(t)$ will be much larger when the driving frequency $\omega = 0.95\omega_0$ than for the case when $\omega = 0.5\omega_0$. Formally, when $\omega = \omega_0$ the amplitude will be infinitely large. We call this resonance. The unphysical situation with an infinite amplitude will be removed when damping is introduced, however, the amplitude may still be very large at resonance.

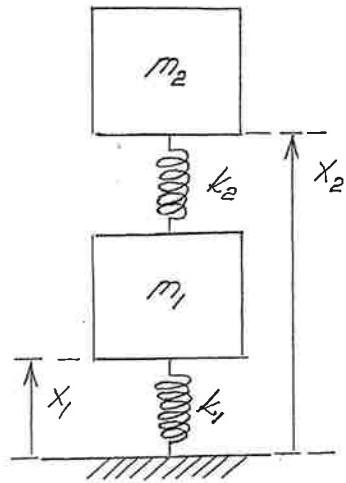
From this we may draw the conclusion

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that it is essential not to excite the structure with a frequency in the neighbourhood of the eigenfrequency. One of the objectives of structural design may then be to see to it that ω is sufficiently far from ω_0 .

Now if we increase the complexity of the system by introducing yet another mass, connected to the first by a second spring, what will then happen?

The equations of free vibrations for the system are given by:



$$\begin{cases} m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \\ m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0 \end{cases} \quad (I.8)$$

Try to derive these equations! Draw free body diagrams for the two

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bodies and formulate the equations of motion!

using matrix notation these equations may be written on the following format (show it!):

$$\underline{M} \ddot{\underline{x}} + \underline{K} \underline{x} = \underline{0} \quad (I.9)$$

where

$$\underline{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \quad (I.10)$$

is called the mass-matrix and

$$\underline{K} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \quad (I.11)$$

is called the stiffness-matrix. The configuration (position) of the system is represented by the column vector

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$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(I.12)

$\bar{0}$ is the column-vector containing only zeros ("the zero-vector"), i.e.

$$\bar{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(I.13)

By assuming harmonic free vibrations

$$\bar{x} = \bar{a} \sin \omega t \quad (I.14)$$

where $\bar{a} \neq \bar{0}$ is a constant vector, we obtain the following necessary condition for the determination of the frequency ω and the amplitude $\bar{a}$:

$$(-\omega^2 \underline{M} + \underline{K}) \bar{a} = \bar{0} \quad (I.15)$$

A solution $\bar{a} \neq \bar{0}$ will exist if and only if (det = determinant)

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$$\det(-\omega^2 \underline{K} + \underline{M}) = 0$$

(I.16)

This is called the characteristic equation to the (generalized) eigenvalue problem (I.15). It is a second order algebraic equation in the unknown ω^2 :

$$(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2) - k_2^2 = 0 \quad (I.17)$$

with the solutions: (show this!)

$$\omega_0^2 = \frac{1}{2} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \pm \sqrt{\frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \right) \quad (I.18)$$

presenting two eigenfrequencies of the system, $\omega_{0,1}$ and $\omega_{0,2}$. The corresponding amplitude-vectors (or "mode-shape vectors") $\bar{a}_1$ and $\bar{a}_2$ will be found by solving the equations

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$$(-\omega_{0,1}^2 \underline{M} + \underline{K}) \bar{a}_1 = \bar{0} \quad (I.19)$$

$$(-\omega_{0,2}^2 \underline{M} + \underline{K}) \bar{a}_2 = \bar{0}$$

Note that if $\bar{x}(t) = \bar{a} \sin \omega t$ is a solution to (I.9) then $\bar{x}(t) = \bar{a} \cos \omega t$ is a solution as well. The general free vibration of the system may be written:

$$\bar{x}(t) = \bar{a}_1 (A \cos \omega_{0,1} t + B \sin \omega_{0,1} t) + \bar{a}_2 (C \cos \omega_{0,2} t + D \sin \omega_{0,2} t) \quad (I.20)$$

where A, B, C and D are arbitrary constants. These may be determined by the initial data:

$$\bar{x}(0) = \bar{x}_0, \quad \dot{\bar{x}}(0) = \bar{v}_0 \quad (I.21)$$

where $\bar{x}_0$ and $\bar{v}_0$ are prescribed (given) constant vectors. The conditions (I.21)

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may be written:

$$\bar{x}_0 = \bar{a}_1 A + \bar{a}_2 C \quad (I.22)$$

$$\bar{v}_0 = \bar{a}_1 B \omega_{0,1} + \bar{a}_2 D \omega_{0,2}$$

To solve this system of equations we may use the fact that

$$\bar{a}_i^T \underline{M} \bar{a}_j = 0 \quad \text{if } i \neq j \quad (I.23)$$

From this "orthogonality property", and (I.22), it follows that

$$\bar{a}_1^T \underline{M} \bar{x}_0 = \bar{a}_1^T \underline{M} \bar{a}_1 A \quad (I.24)$$

and then

$$A = \frac{\bar{a}_1^T \underline{M} \bar{x}_0}{\bar{a}_1^T \underline{M} \bar{a}_1} \quad (I.25)$$

In a similar way we obtain

$$B = \frac{\bar{a}_1^T \underline{M} \bar{x}_0}{\bar{a}_2^T \underline{M} \bar{a}_2}, \quad C = \frac{\bar{a}_2^T \underline{M} \bar{v}_0}{\omega_{0,1} \bar{a}_1^T \underline{M} \bar{a}_1} \quad (I.26)$$

$$D = \frac{\bar{a}_2^T \underline{M} \bar{v}_0}{\omega_{0,2} \bar{a}_2^T \underline{M} \bar{a}_2}$$

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and then the free vibration may be written:

$$\bar{x}(t) = \sum_{i=1}^2 \left\{ \frac{\bar{a}_i \bar{a}_i^T M}{\bar{a}_i^T M \bar{a}_i} \bar{x}_0 \cos \omega_{0,i} t + \frac{\bar{a}_i \bar{a}_i^T M}{\omega_{0,i} \bar{a}_i^T M \bar{a}_i} \bar{v}_0 \sin \omega_{0,i} t \right\} \quad (I.27)$$

For a vibrating system with:

$$m_1 = 20 \text{ kg}, \quad m_2 = 30 \text{ kg}, \quad k_1 = 20 \text{ kNm}^{-1}$$

$$k_2 = 30 \text{ kNm}^{-1}$$

We get the eigenfrequencies:

$$\omega_{01} = 24.85^{-1} \quad (3.9 \text{ Hz})$$

$$\omega_{02} = 69.95^{-1} \quad (11.1 \text{ Hz})$$

The mode-shape vectors are

$$\bar{a}_1 = \begin{pmatrix} 1 \\ 1.26 \end{pmatrix}, \quad \bar{a}_2 = \begin{pmatrix} 1 \\ -1.59 \end{pmatrix}$$

Using this in formula (I.27) together

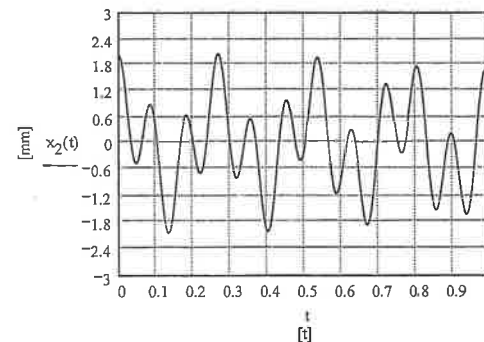
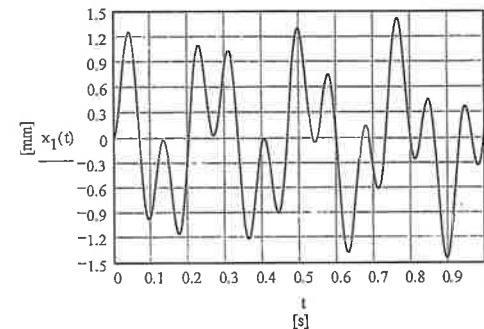
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with the initial data

$$\bar{x}_0 = \begin{pmatrix} 0 \\ 0.002 \end{pmatrix} \quad [\text{m}]$$

$$\bar{v}_0 = \begin{pmatrix} 0.01 \\ 0 \end{pmatrix} \quad [\text{ms}^{-1}]$$

We get the following free vibration.



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Now what happens if we apply external forces on our system?

We consider a harmonic excitation with

$$P_1 = \hat{P}_1 \sin \omega t \quad (I.28)$$

$$P_2 = \hat{P}_2 \sin \omega t$$

where $\hat{P}_1, \hat{P}_2$ are

the amplitudes and ω is the frequency of the excitation. We summarize this in the vector:

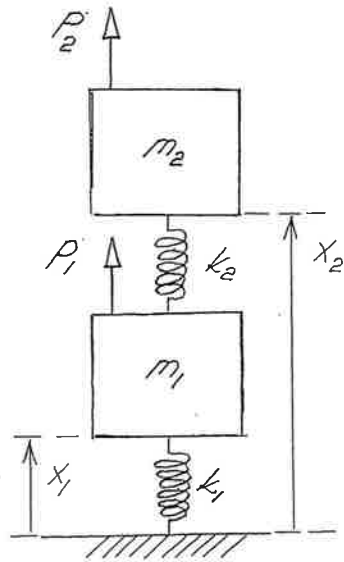
$$\bar{P}(t) = \hat{\bar{P}} \sin \omega t$$

(I.29)

$$\bar{P} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \quad \hat{\bar{P}} = \begin{pmatrix} \hat{P}_1 \\ \hat{P}_2 \end{pmatrix}$$

The equations of motion now take the form:

$$\underline{M} \ddot{\bar{X}} + \underline{K} \bar{X} = \bar{P} = \hat{\bar{P}} \sin \omega t \quad (I.30)$$



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A solution to (I.30) is given by:

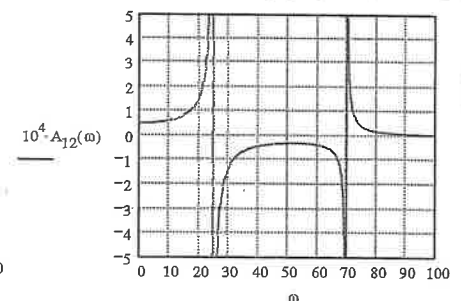
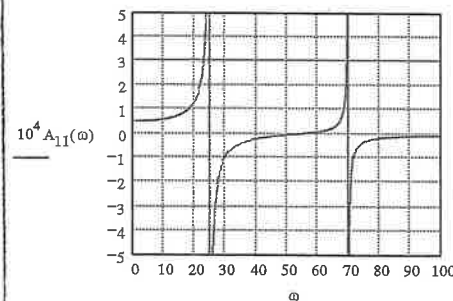
$$\bar{X}(t) = \underline{E}(\omega) \bar{P} \quad (I.31)$$

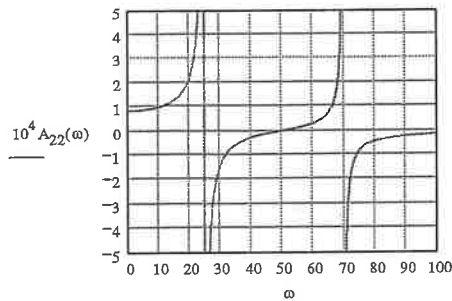
where the matrix $\underline{X}$:

$$\underline{E}(\omega) = \sum_{i=1}^2 \frac{\bar{a}_i \bar{a}_i^T}{(\omega_{0i}^2 - \omega^2) \mu_i} \quad (I.32)$$

$$\mu_i = \bar{a}_i^T \underline{M} \bar{a}_i$$

is the Frequency response function of the system. In the figures below the matrix elements of $\underline{E}$ are presented. Note that $F_{12} = F_{21}$.





The two resonances at $\omega = \omega_{0,1} \approx 24.85^{-1}$ and $\omega = \omega_{0,2} \approx 69.95^{-1}$ are clearly recognizable as the peaks in the graphs.

Frequency response functions and, more general, transfer functions are important tools for analyzing mechanical vibrations. In this course we will give these concepts, and many more, a thorough study.

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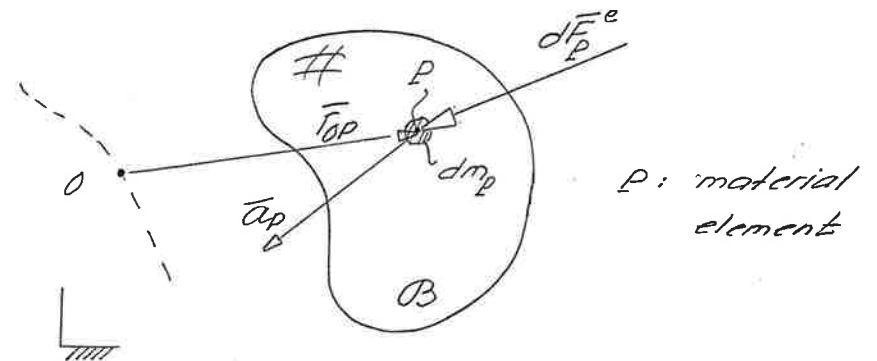
LECTURE 1

EQUATIONS OF MOTION

Euler's equations
Virtual work
d'Alembert's principle

EQUATIONS OF MOTION

1. The Euler equations



"Master laws" (Euler 1775)

$$\vec{F}^e = \int_B d\vec{F}_p^e = \int_B \vec{a}_p dm_p \quad (I)$$

$$\vec{M}_O^e = \int_B \vec{r}_{OP} \times d\vec{F}_p^e = \int_B \vec{r}_{OP} \times \vec{a}_p dm_p \quad (II)$$

$d\vec{F}_p^e$: external forces.

$\vec{a}_p$: mass element acceleration.

dm_p : mass element.

The interpretation of the force system $\{d\bar{F}_p^e\}$ and

$$\left| \int_B d\bar{F}_p^e, \int_B \bar{F}_{op} \times d\bar{F}_p^e \right|$$

will depend on the actual mechanical system, (particles, rigid bodies, continua).

Note that (in general):

$$\left| d\bar{F}_p^e \neq \bar{a}_p dm_p \right|$$

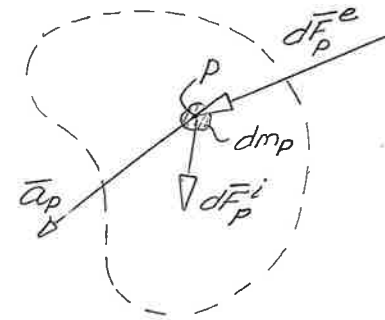
The difference:

$$\left| d\bar{F}_p^i = \bar{a}_p dm_p - d\bar{F}_p^e \right|$$

is called the internal force

$$\left\| d\bar{F}_p^e + d\bar{F}_p^i = \bar{a}_p dm_p \right. \quad (1.1)$$

2. The accelerating force.



"Master law" for material element

$$\left\| d\bar{F}_p = d\bar{F}_p^e + d\bar{F}_p^i = \bar{a}_p dm_p \right. \quad (III)$$

$d\bar{F}_p^i$: internal forces

(III) $\Rightarrow$

$$\left| \int_B d\bar{F}_p^e + \int_B d\bar{F}_p^i = \int_B \bar{a}_p dm_p \right. \quad (2.1)$$

$\Rightarrow$ (using (I))

$$\left\| \bar{F}^i = \int_B d\bar{F}_p^i = \bar{\sigma} \right. \quad (IV)$$

(III) $\Rightarrow$

$$\bar{r}_{op} \times d\bar{F}_p^e + \bar{r}_{op} \times d\bar{F}_p^i = \bar{r}_{op} \times \bar{a}_p dm_p$$

 $\Rightarrow$

$$\int_B \bar{r}_{op} \times d\bar{F}_p^e + \int_B \bar{r}_{op} \times d\bar{F}_p^i = \int_B \bar{r}_{op} \times \bar{a}_p dm_p \quad (2.2)$$

 $\Rightarrow$ (using (II)):

$$\bar{M}_0^i = \int_B \bar{r}_{op} \times d\bar{F}_p^i = \bar{0} \quad (V)$$

"The system of internal forces is a null-system".

We find that:

$$(I) + (II) + (III) \Rightarrow (IV) + (V)$$

$$(III) + (IV) + (V) \Rightarrow (I) + (II)$$

$d\bar{F} = d\bar{F}^e + d\bar{F}^i$: accelerating force

Note that

$$\bar{F} = \int_B d\bar{F}_p = \int_B \bar{a}_p dm_p \quad (2.3)$$

$$\bar{M}_0 = \int_B \bar{r}_{op} \times d\bar{F}_p = \int_B \bar{r}_{op} \times \bar{a}_p dm_p$$

Specific* accelerating forces:

$$\bar{f}_p = \frac{d\bar{F}_p}{dm_p}, \quad \bar{f}_p^e = \frac{d\bar{F}_p^e}{dm_p}, \quad \bar{f}_p^i = \frac{d\bar{F}_p^i}{dm_p} \quad (2.5)$$

$$\bar{f}_p = \bar{f}_p^e + \bar{f}_p^i$$

$$\bar{f}_p = \bar{a}_p$$

$$\int_B \bar{f}_p^i dm_p = \bar{0} \quad (2.6)$$

$$\int_B \bar{r}_{op} \times \bar{f}_p^i dm_p = \bar{0}$$

* mass-specific

We have:

$$\bar{F}^e = \int_B d\bar{F}_p^e = \int_B \bar{F}_p dm_p \quad (2.7)$$

$$\bar{M}_O^e = \int_B \bar{r}_{Op} \times d\bar{F}_p^e = \int_B \bar{r}_{Op} \times \bar{F}_p dm_p$$

The system of external forces $\{d\bar{F}_p^e\}$ is equipollent^{*} to the system of accelerating forces $\{\bar{F}_p\}$.

Two systems of forces $\{d\bar{F}_p\}$ and $\{d\bar{G}_p\}$ are said to be equipollent if they have the same force sum and the same moment sum with respect to some point O , i.e.

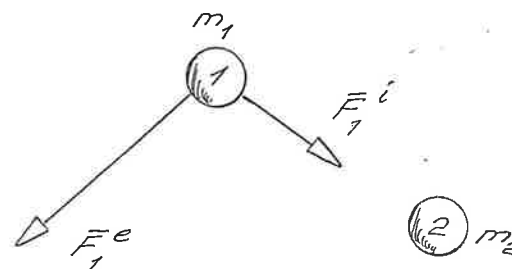
$$\int_B d\bar{F}_p = \int_B d\bar{G}_p \quad (2.8)$$

$$\int_B \bar{r}_{Op} \times d\bar{F}_p = \int_B \bar{r}_{Op} \times d\bar{G}_p$$

* equipollent

Ex.1: Two particle system

Particles with masses m_1 and m_2



$$\bar{F}_1^e + \bar{F}_1^i = \bar{a}_1 m_1 \quad \Rightarrow$$

$$\bar{F}_2^e + \bar{F}_2^i = \bar{a}_2 m_2$$

$$\bar{F}_1 = \frac{1}{m_1} (\bar{F}_1^e + \bar{F}_1^i)$$

$$\bar{F}_2 = \frac{1}{m_2} (\bar{F}_2^e + \bar{F}_2^i)$$

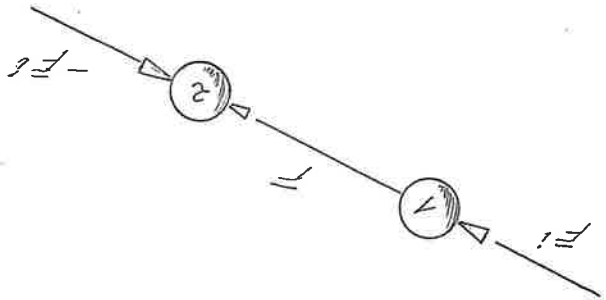
Internal forces: $\{\bar{F}_1^i, \bar{F}_2^i\}$

$$\bar{F}_1^i + \bar{F}_2^i = \bar{0}$$

$$\bar{r}_1 \times \bar{F}_1^i + \bar{r}_2 \times \bar{F}_2^i = \bar{0} \quad \Rightarrow$$

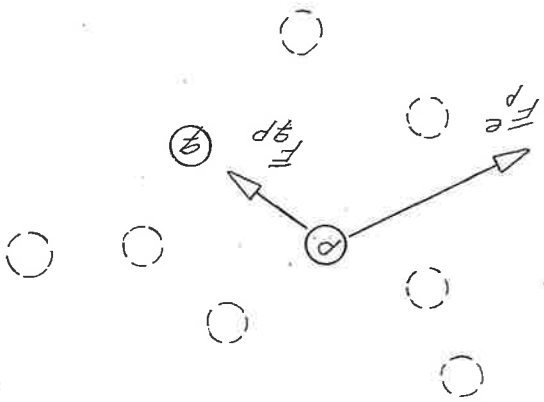
$$\begin{aligned} \vec{F}_1' &= -\vec{F}_2' = \vec{F}_1 \\ (\vec{r}_2 - \vec{r}_1) \times \vec{F}_1' &= \vec{0} \end{aligned}$$

$$\vec{F}_1' \parallel \vec{r}_2 - \vec{r}_1$$



#

Ex 2: Many particle system.



#

$$\begin{aligned} \vec{F}_p' + \vec{F}_p^c &= \vec{a}_p m_p \\ \vec{F}_p' &= \sum_{q=1}^n \vec{F}_{qp} \quad , \quad n = \text{number of particles} \\ \|\vec{F}_p' &= \frac{1}{m_p} (\vec{F}_p^c + \sum_{q=1}^n \vec{F}_{qp}) \end{aligned}$$

Note that:

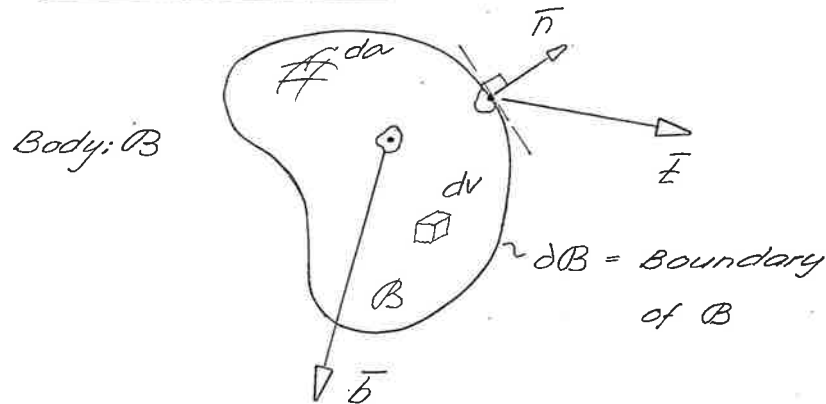
$$\vec{F}_{qp} = -\vec{F}_{pq} \quad , \quad \vec{F}_{qp} \parallel \vec{r}_q - \vec{r}_p$$

and then:

$$\begin{aligned} \sum_{p=1}^n \vec{F}_p' m_p &= \sum_{p,q=1}^n \vec{F}_{pq} = \vec{0} \\ \sum_{p=1}^n \vec{F}_p \times \vec{r}_p &= \sum_{p=1}^n \vec{F}_p \times \vec{r}_p = \sum_{p=1}^n \vec{F}_p \times \vec{r}_p = \vec{0} \end{aligned}$$

Ex.3. Continuum mechanical system.

(This is optional)



$$\vec{F}^e = \int_{\partial B} \vec{F} da + \int_B \vec{b} dv$$

$$\vec{M}_o^e = \int_{\partial B} \vec{r} \times \vec{F} da + \int_B \vec{r} \times \vec{b} dv$$

$\vec{F}$: contact force (per unit area)

$\vec{b}$: body force (per unit volume)

$$\vec{F} = \underline{T} \vec{n}$$

$\underline{T}$: stress tensor

In the continuum mechanics course it is shown that the Master laws I & II will imply the following conditions

$$\rho \vec{a} = \text{div } \underline{T} + \vec{b}$$

$$\underline{T}^T = \underline{T}$$

ρ : mass density

Then we get the accelerating force:

$$\vec{F} = \frac{1}{\rho} (\text{div } \underline{T} + \vec{b})$$

The sum of the external forces:

$$\vec{F}^e = \int_{\partial B} \underline{T} \vec{n} da + \int_B \vec{b} dv = \int_B (\text{div } \underline{T} + \vec{b}) dv$$

$$\vec{b} dv = \int_B d\vec{F}^e$$

The sum of the moments of the

external forces

$$\bar{M}_0^e = \int_{\partial B} \bar{F} \times \bar{I} \bar{n} da + \int_B \bar{F} \times \bar{b} dv =$$

$$\int_B \bar{F} \times (\text{div} \bar{I} + \bar{b}) dv = \int_B \bar{F} \times d\bar{F}^e$$

We may then conclude that

$$d\bar{F}^e = (\text{div} \bar{I} + \bar{b}) dv - d\bar{I}$$

where $\{d\bar{I}\}$ is a null system, i.e.

$$\int_B d\bar{I} = \bar{0}, \quad \int_B \bar{F} \times d\bar{I} = \bar{0}$$

and then

$$d\bar{F}^e + d\bar{I} = (\text{div} \bar{I} + \bar{b}) dv = \bar{a} dm =$$

$$d\bar{F}$$

$\{\bar{I}\}$ is thus the system of internal forces.

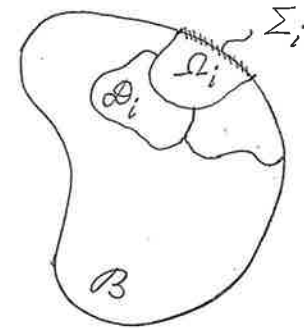
The internal force system (optional)

Let $\Omega_1, \dots, \Omega_n, \mathcal{Q}_1, \dots, \mathcal{Q}_m$

$$\partial\Omega_i \cap \partial B = \Sigma_i$$

$$\partial\mathcal{Q}_i \cap \partial B = \emptyset$$

be an arbitrary partitioning of B



a partition
of B

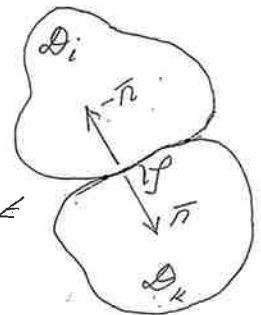
The sum of the internal forces:

$$\bar{I} = \int_B d\bar{I} = \sum_{i=1}^n \int_{\partial\Omega_i \setminus \Sigma_i} \bar{I} \bar{n} da +$$

$$\sum_{i=1}^m \int_{\partial\mathcal{Q}_i} \bar{I} \bar{n} da = \bar{0}$$

since for subbodies in contact

$$\int_{\partial\mathcal{Q}_i \cap \mathcal{J}} \bar{I} \bar{n} da + \int_{\partial\mathcal{Q}_k \cap \mathcal{J}} \bar{I} (-\bar{n}) da = \bar{0}$$



$$\mathcal{J} = \partial\mathcal{Q}_i \cap \partial\mathcal{Q}_k$$

This also follows from:

$$\sum_{i=1}^n \int_{\partial \Omega_i} I \bar{n} da + \sum_{i=1}^m \int_{\partial \Omega_i} I \bar{n} da +$$

$$\sum_{i=1}^n \int_{\partial \Omega_i} I \bar{n} da + \sum_{i=1}^m \int_{\partial \Omega_i} I \bar{n} da - \sum_{i=1}^n \int_{\Sigma_i} I \bar{n} da$$

$$= \sum_{i=1}^n \int_{\Omega_i} \operatorname{div} I dv + \sum_{i=1}^m \int_{\Omega_i} \operatorname{div} I dv -$$

$$\int_{\partial \Omega} I \bar{n} da = \int_{\Omega} \operatorname{div} I dv - \int_{\partial \Omega} I \bar{n} da = \bar{0}$$

We also have

$$\int_{\Omega} \bar{r} \times d\bar{I} = \bar{0}$$

and the internal force system $\{d\bar{I}\}$ for any partitioning of Ω is a null system.

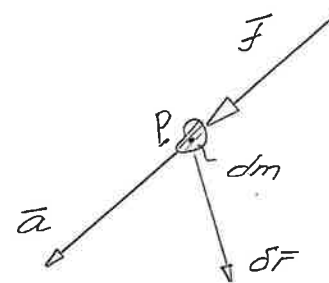
3. Virtual work

Master law for mass elements:

$$\| \bar{f} = \bar{a} \Leftrightarrow \bar{f} - \bar{a} = \bar{0} \quad (3.1)$$

$\bar{f}$: specific accelerating force

$\bar{a}$: acceleration



$$\bar{f} = \bar{f}^e + \bar{f}^i$$

$$\delta \bar{f} = \delta \bar{f}(\bar{f}):$$

vector-field on the body

$$\bar{f} - \bar{a} = \bar{0}$$

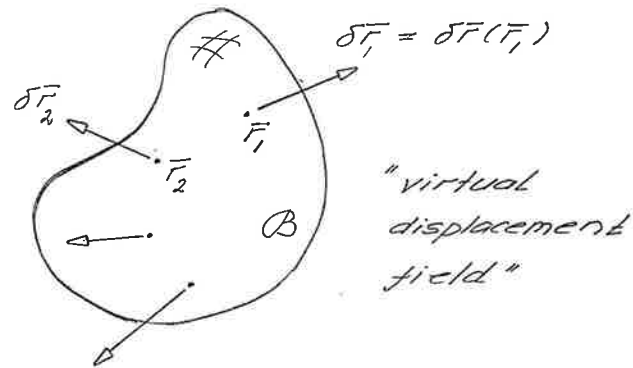


$$(\bar{f} - \bar{a}) \cdot \delta \bar{f} = 0, \quad \forall \delta \bar{f}$$



$$\| \int_{\Omega} (\bar{f} - \bar{a}) \cdot \delta \bar{f} dm = 0, \quad \forall \delta \bar{f} = \delta \bar{f}(\bar{f}) \quad (3.2)$$

$\delta \vec{r} = \delta \vec{r}(\vec{r})$: virtual displacement field.



By a constraint we mean a functional relation between material positions in the system:

$$G(\vec{r}_1, \vec{r}_2, \dots) = 0 \quad (3.3)$$

For a virtual displacement, respecting the constraint, we have:

$$\delta G = \frac{\partial G}{\partial \vec{r}_1} \cdot \delta \vec{r}_1 + \frac{\partial G}{\partial \vec{r}_2} \cdot \delta \vec{r}_2 + \dots = 0 \quad (3.4)$$

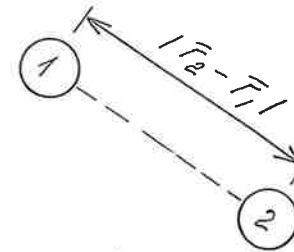
which implies a relation between:

$$\delta \vec{r}_1, \delta \vec{r}_2, \dots$$

Ex. 4: For a two-particle system we may impose the constraint:

$$G(\vec{r}_1, \vec{r}_2) = |\vec{r}_2 - \vec{r}_1|^2 - C = 0$$

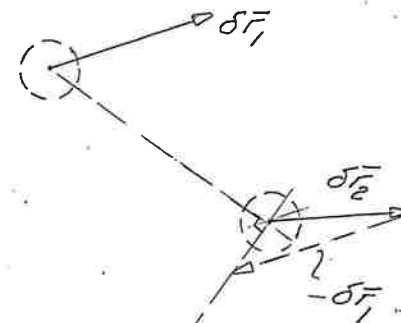
$C = \text{constant}$.



Then we have: $\frac{\partial G}{\partial \vec{r}_1} = 2(\vec{r}_1 - \vec{r}_2)$, $\frac{\partial G}{\partial \vec{r}_2} = 2(\vec{r}_2 - \vec{r}_1)$

$$\delta G = 2(\vec{r}_1 - \vec{r}_2) \cdot \delta \vec{r}_1 + 2(\vec{r}_2 - \vec{r}_1) \cdot \delta \vec{r}_2 =$$

$$2(\vec{r}_2 - \vec{r}_1) \cdot (\delta \vec{r}_2 - \delta \vec{r}_1) = 0$$



Let us split the accelerating force

$$\vec{F} = \vec{F}_a + \vec{F}_c \quad (3.5)$$

$\vec{F}_a$: active forces.

$\vec{F}_c$: forces of constraints

We assume that

$$\delta W_c = \int_B \vec{F}_c \cdot \delta \vec{r} \, dm = 0 \quad (3.6)$$

$\forall \delta \vec{r} = \delta \vec{r}(\vec{r})$ satisfying the constraints. The virtual work of the forces of constraints is zero.

Then (the principle of d'Alembert)

$$\int_B (\vec{F}_a - \vec{a}) \cdot \delta \vec{r} \, dm = 0 \quad (VII)$$

$\forall \delta \vec{r} = \delta \vec{r}(\vec{r})$ satisfying the constraints.

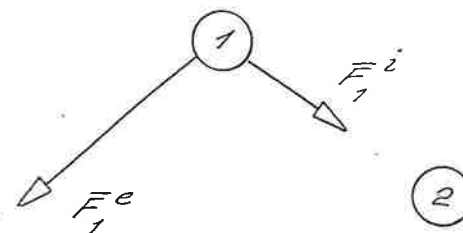
$$\delta W_a = \int_B \vec{F}_a \cdot \delta \vec{r} \, dm \quad (3.7)$$

is the virtual work of active forces.

$$\delta W_a = \int_B \vec{a} \cdot \delta \vec{r} \, dm \quad (3.8)$$

Ex. 5: Two particle system with internal constraint.

$$|\vec{r}_2 - \vec{r}_1|^2 - c^2 = 0$$



$$\delta W_c = \vec{F}_1^i \cdot \delta \vec{r}_1 + \vec{F}_2^i \cdot \delta \vec{r}_2 =$$

$$\vec{F}^i \cdot (\delta \vec{r}_1 - \delta \vec{r}_2) = 0$$

since $\vec{F}^i \parallel (\vec{r}_1 - \vec{r}_2)$ and

$$(\vec{r}_1 - \vec{r}_2) \perp (\delta \vec{r}_1 - \delta \vec{r}_2)$$

Thus the internal forces $\{\vec{F}_1^i, \vec{F}_2^i\}$

are forces of constraints and the external forces $\{\bar{F}_1^e, \bar{F}_2^e\}$ are active.

According to d'Alembert's principle:

$$\bar{F}_1^e \cdot \delta \bar{r}_1 + \bar{F}_2^e \cdot \delta \bar{r}_2 = \bar{a}_1 \cdot \delta \bar{r}_1 m_1 + \bar{a}_2 \cdot \delta \bar{r}_2 m_2$$

Selecting $\delta \bar{r}_1 = \delta \bar{r}_2 = \bar{u}$

$$\bar{F}_1^e \cdot \bar{u} + \bar{F}_2^e \cdot \bar{u} = \bar{a}_1 \cdot \bar{u} m_1 + \bar{a}_2 \cdot \bar{u} m_2, \quad \forall \bar{u}$$

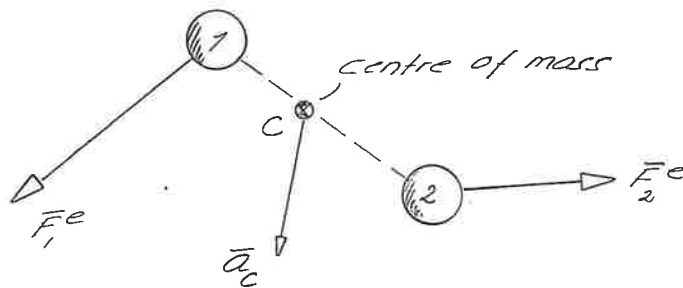
$\Rightarrow$

$$\bar{F}_1^e + \bar{F}_2^e = \bar{a}_1 m_1 + \bar{a}_2 m_2 = \bar{a}_c m$$

$$\bar{a}_c = \frac{1}{m} (\bar{a}_1 m_1 + \bar{a}_2 m_2)$$

$$m = m_1 + m_2$$

$$c = \text{centre of mass}; \quad \bar{r}_c = \frac{1}{m} (\bar{r}_1 m_1 + \bar{r}_2 m_2)$$



#

We may introduce the following split of active forces:

$$\bar{F}_a = \bar{F}_a^e + \bar{F}_a^i \quad (3.9)$$

$\bar{F}_a^e$: external part of active forces

$\bar{F}_a^i$: internal part of active forces

and the associated virtual work:

$$\delta W_a^e = \int_B \bar{F}_a^e \cdot \delta \bar{r} \, dm \quad (3.10)$$

$$\delta W_a^i = \int_B \bar{F}_a^i \cdot \delta \bar{r} \, dm$$

d'Alembert's principle:

$$\delta W_a^e + \delta W_a^i = \int_B \bar{a} \cdot \delta \bar{r} \, dm \quad (3.11)$$

$\forall$ virtual displacement fields $\delta \bar{r}$ respecting the constraints.

A rigid virtual displacement field is defined by:

$$\delta \bar{r} = \delta \bar{r}(\bar{r}) = \bar{u} + \bar{W} \times \bar{r} \quad (3.12)$$

$\bar{u}$, $\bar{W}$ constant vectors.

Note that in this case we have

$$\begin{aligned} \delta |\bar{r}_1 - \bar{r}_2|^2 &= 2(\bar{r}_1 - \bar{r}_2) \cdot (\delta \bar{r}_1 - \delta \bar{r}_2) = \\ 2(\bar{r}_1 - \bar{r}_2) \cdot (\bar{W} \times (\bar{r}_1 - \bar{r}_2)) &= 0 \end{aligned} \quad (3.13)$$

Assume that the rigid virtual displacement field respects the constraints. Then

$$\begin{aligned} 0 &= \int_B \bar{F}_c \cdot \delta \bar{r} \, dm = \int_B \bar{F}_c \cdot (\bar{u} + \bar{W} \times \bar{r}) \, dm = \\ &(\int_B \bar{F}_c \, dm) \cdot \bar{u} + (\int_B \bar{r} \times \bar{F}_c \, dm) \cdot \bar{W} \end{aligned} \quad (3.14)$$

If this applies $\forall \bar{u}, \bar{W}$ then $\{\bar{F}_c\}$ is a null system, i.e.

$$\int_B \bar{F}_c \, dm = \bar{0}, \quad \int_B \bar{r} \times \bar{F}_c \, dm = \bar{0} \quad (3.15)$$

From d'Alemberts principle:

$$\int_B \bar{F}_a \, dm = \int_B \bar{a} \, dm \quad (3.16)$$

$$\int_B \bar{r} \times \bar{F}_a \, dm = \int_B \bar{r} \times \bar{a} \, dm$$

If we put

$$\bar{F} = \bar{F}_a + \bar{F}_c = \bar{F}^e + \bar{F}^i \quad (3.17)$$

where

$$\int_B \bar{F}^i \, dm = \bar{0}, \quad \int_B \bar{r} \times \bar{F}^i \, dm = \bar{0} \quad (3.18)$$

then from (3.15):

$$\int_B \bar{F}^e \, dm = \int_B (\bar{F}_a + \bar{F}_c - \bar{F}^i) \, dm =$$

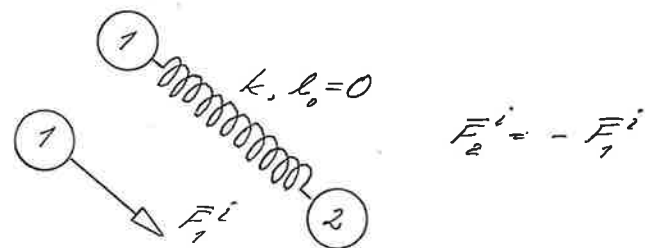
$$\int_B \bar{F}_a \, dm = \int_B \bar{a} \, dm \quad (3.19)$$

$$\int_B \bar{r} \times \bar{F}^e \, dm = \int_B \bar{r} \times \bar{a} \, dm \quad (3.20)$$

which means that we recover the Euler equations. We may formulate this as a

Theorem Assuming the validity of the principle of d'Alembert and that the system of internal forces is a null system and finally, that all rigid displacement fields respect the constraint conditions, then the Euler equations (I)-(II) are valid.

Ex 6: Two particle system with linear elastic spring.



$$\delta W_a^i = \bar{F}_1^i \cdot \delta \bar{r}_1 + \bar{F}_2^i \cdot \delta \bar{r}_2 = \bar{F}^i \cdot (\delta \bar{r}_1 - \delta \bar{r}_2)$$

Assume (linear elastic internal force)

$$\bar{F}^i = k(\bar{r}_2 - \bar{r}_1)$$

Then

$$\delta W_a^i = k(\bar{r}_2 - \bar{r}_1) \cdot (\delta \bar{r}_1 - \delta \bar{r}_2) =$$

$$- \delta \left(\frac{1}{2} k |\bar{r}_2 - \bar{r}_1|^2 \right) = - \delta V_e$$

where

$$V_e = V_e(\bar{r}_1, \bar{r}_2) = \frac{1}{2} k |\bar{r}_2 - \bar{r}_1|^2$$

is the elastic potential. d'Alembert $\Rightarrow$

$$\delta W_a^e = \bar{a}_1 \cdot \delta \bar{r}_1 m_1 + \bar{a}_2 \cdot \delta \bar{r}_2 m_2 + \delta V_e$$

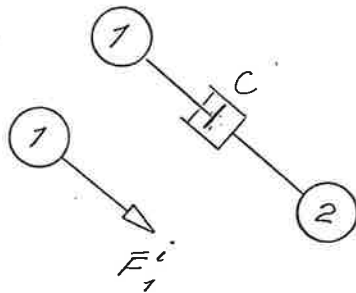
Ex 7: Many particle system with linear elastic springs.

the elastic potential:

$$V_e = V_e(\vec{r}_1, \vec{r}_2, \dots) = \sum_{p < q} \frac{k_{pq}}{2} |\vec{r}_p - \vec{r}_q|^2$$

#

Ex 8: Two particle system with linear viscous damper



$$\vec{F}_2^i = -\vec{F}_1^i$$

$$\delta W_a^i = \vec{F}_1^i \cdot \delta \vec{r}_1 + \vec{F}_2^i \cdot \delta \vec{r}_2 = \vec{F}_1^i \cdot (\delta \vec{r}_1 - \delta \vec{r}_2)$$

Assume (linear viscous internal force)

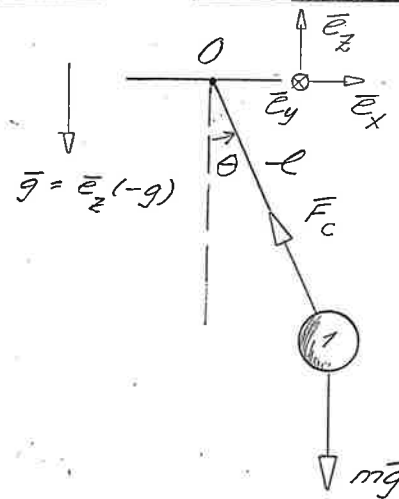
$$\vec{F}^i = C(\vec{v}_2 - \vec{v}_1), \quad \vec{v}_2 = \dot{\vec{r}}_2, \quad \vec{v}_1 = \dot{\vec{r}}_1$$

Then

$$\delta W_a^i = -C(\vec{v}_2 - \vec{v}_1) \cdot (\delta \vec{r}_2 - \delta \vec{r}_1)$$

#

Ex. 9: Simple planar pendulum



Constraint:

$$|\vec{r}_1|^2 - l^2 = 0$$

$$\vec{r}_1 \cdot \delta \vec{r}_1 = 0$$

$$\vec{F}_c \parallel \vec{r}_1 \Rightarrow$$

$$\vec{F}_c \cdot \delta \vec{r}_1 = 0$$

$$\delta W_c = \vec{F}_c \cdot \delta \vec{r}_1 = 0$$

$$\delta W_a = m\vec{g} \cdot \delta \vec{r}_1 = -mg \delta \underbrace{(\vec{e}_z \cdot \vec{r}_1)}_{=z} = -mg \delta z$$

$$z = -l \cos \theta \Rightarrow \delta z = l \sin \theta \delta \theta$$

$$\delta W_a = -mg l \sin \theta \delta \theta = \delta W_a^c$$

$$\vec{a} = \ddot{\vec{r}}_1 = \frac{d^2}{dt^2} (\vec{e}_x l \sin \theta + \vec{e}_z (-l \cos \theta))$$

$$= \vec{e}_x (l \ddot{\theta} \cos \theta - l \dot{\theta}^2 \sin \theta) + \vec{e}_z (l \ddot{\theta} \sin \theta + l \dot{\theta}^2 \cos \theta)$$

$$\delta \vec{r}_1 = \vec{e}_x l \cos \theta \delta \theta + \vec{e}_z l \sin \theta \delta \theta$$

is respecting the constraint

d'Alembert's principle $\Rightarrow$

$$\delta W_a = \bar{a} \cdot \delta \vec{r}_m$$

A $\delta \vec{r}_i$ respecting the constraints

$$-mg \sin \theta \delta \theta = \vec{r}_m'' \cdot \delta \vec{r}_\theta, \quad A \delta \theta$$

$\Rightarrow$

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0$$

#

LECTURE 2

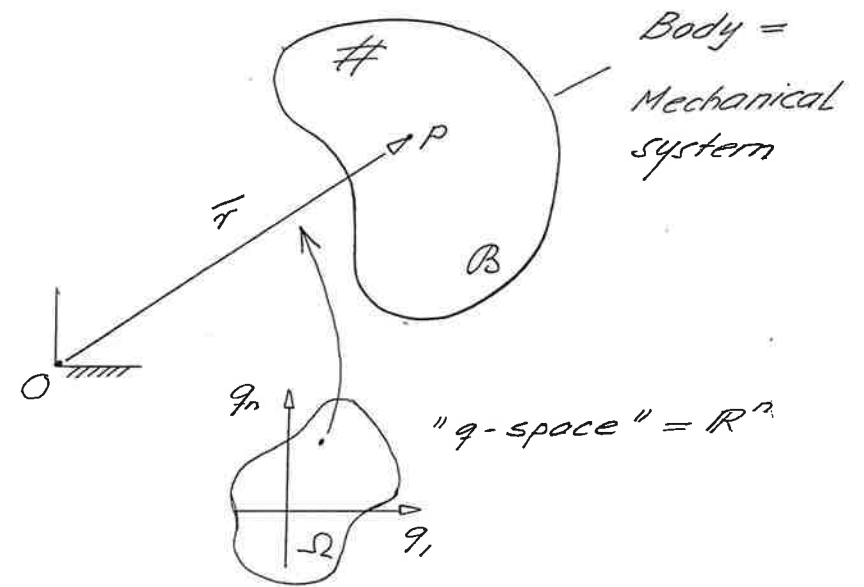
EQUATIONS OF MOTION

Lagrange's equations
The structure of the kinetic energy
Conservative forces
Non-conservative forces

4. Lagrange's equations

Generalized coordinates:

$$\begin{aligned} & q_1, q_2, \dots, q_n \quad ; \quad q_i = q_i(t) \\ & \bar{F} = \bar{F}(t, q_1, \dots, q_n; P) = \bar{F}_{OP} \end{aligned} \quad (4.1)$$



Ω an open set in q -space.

Virtual displacement:

$$\delta \bar{F} = \sum_{i=1}^n \frac{\partial \bar{F}}{\partial q_i} \delta q_i \quad (4.2)$$

$$\delta q = (\delta q_1, \dots, \delta q_n) \in \mathbb{R}^n$$

The generalized coordinates $q = (q_1, \dots, q_n) \in \Omega \subseteq \mathbb{R}^n$ are sometimes referred to as configuration coordinates. The requirements on the q -coordinates are: that they will, together with time t , through the expression (4.1), determine the position vector for every material point $P \in \mathcal{B}$. Furthermore we assume that the function (4.1), for all $P \in \mathcal{B}$, is defined in a common $(n+1)$ -dimensional $(t, \bar{q})$ -region where all partial derivatives exist and are continuous.

The q -coordinate system is said to be regular if

$$\delta \bar{r} = \sum_{i=1}^n \frac{\partial \bar{r}}{\partial q_i} \delta q_i = \bar{0}, \quad \forall P \in \mathcal{B} \iff \delta q_1 = \delta q_2 = \dots = \delta q_n = 0 \quad (4.3)$$

The q -coordinate system is said to be scleronomous if the function (4.1) does not contain time t explicitly, i.e.

$$\left| \frac{\partial \bar{r}}{\partial t} = \bar{0}. \quad (4.4) \right|$$

The virtual work of the active forces $\bar{f}_a$ may now be written:

$$\left| \delta W_a = \int_{\mathcal{B}} \bar{f}_a \cdot \delta \bar{r} \, dm = \sum_{i=1}^n \left(\int_{\mathcal{B}} \bar{f}_a \cdot \frac{\partial \bar{r}}{\partial q_i} \, dm \right) \delta q_i \right|$$

and for the accelerations:

$$\left| \int_{\mathcal{B}} \bar{a} \cdot \delta \bar{r} = \sum_{i=1}^n \left(\int_{\mathcal{B}} \bar{a} \cdot \frac{\partial \bar{r}}{\partial q_i} \, dm \right) \delta q_i \right|$$

d'Alemberts principle:

$$\left| \delta W_a = \int_{\mathcal{B}} \bar{a} \cdot \delta \bar{r} \, dm, \quad \forall \delta q \in \mathbb{R}^n \right|$$

is then equivalent to

$$\left| \int_{\mathcal{B}} \bar{f}_a \cdot \frac{\partial \bar{r}}{\partial q_i} \, dm = \int_{\mathcal{B}} \bar{a} \cdot \frac{\partial \bar{r}}{\partial q_i} \, dm, \quad i=1, \dots, n \right| \quad (4.5)$$

Now we have:

$$\vec{a}: \frac{\partial \vec{r}}{\partial q_i} = \dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i} = \frac{d\vec{r}}{dt} \cdot \frac{\partial \vec{r}}{\partial q_i} =$$

$$\frac{d}{dt} \left(\dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i} \right) - \dot{\vec{r}} \cdot \frac{d}{dt} \frac{\partial \vec{r}}{\partial q_i} \quad (4.6)$$

But

$$\dot{\vec{r}} = \frac{\partial \vec{r}}{\partial t} + \sum_{i=1}^n \frac{\partial \vec{r}}{\partial q_i} \dot{q}_i = \dot{\vec{r}}(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$$

$$\Rightarrow \underline{\frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} = \frac{\partial \vec{r}}{\partial q_i}} \quad i = 1, \dots, n \quad (4.7)$$

and consequently

$$\left| \dot{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial q_i} = \dot{\vec{r}} \cdot \frac{d\vec{r}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \left(\frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{2} \right) \right. \quad (4.8)$$

Furthermore

$$\left| \frac{d}{dt} \frac{\partial \vec{r}}{\partial q_i} = \frac{\partial^2 \vec{r}}{\partial t \partial q_i} + \sum_{j=1}^n \frac{\partial^2 \vec{r}}{\partial q_j \partial q_i} \dot{q}_j = \right.$$

$$\frac{\partial^2 \vec{r}}{\partial q_i \partial t} + \sum_{j=1}^n \frac{\partial^2 \vec{r}}{\partial q_j \partial q_i} \dot{q}_j = \frac{\partial}{\partial q_i} \left(\frac{d\vec{r}}{dt} + \sum_{j=1}^n \frac{\partial \vec{r}}{\partial q_j} \dot{q}_j \right) =$$

$$\left| \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} \right. \quad (4.9)$$

and then

$$\dot{\vec{r}} \cdot \frac{d}{dt} \frac{\partial \vec{r}}{\partial q_i} = \dot{\vec{r}} \cdot \frac{\partial \dot{\vec{r}}}{\partial \dot{q}_i} = \frac{\partial}{\partial \dot{q}_i} \left(\frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{2} \right) \quad (4.10)$$

Insertion of (4.8) and (4.10) into the right hand side of (4.6)

$$\int_B \vec{f}_a \cdot \frac{\partial \vec{r}}{\partial q_i} dm = \int_B \left(\frac{d}{dt} \frac{\partial}{\partial \dot{q}_i} \left(\frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{2} \right) - \frac{\partial}{\partial q_i} \left(\frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{2} \right) \right) dm =$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i}, \quad i = 1, \dots, n \quad (4.11)$$

where T is the kinetic energy

$$\left\| T = T(t, q, \dot{q}) = \int_B \frac{\dot{\vec{r}} \cdot \dot{\vec{r}}}{2} dm \right. \quad (4.12)$$

We may now formulate Lagrange's equations:

$$\left\| Q_i = \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} \quad i = 1, \dots, n \right.$$

$$\left\| Q_i = \int_B \vec{f}_a \cdot \frac{\partial \vec{r}}{\partial q_i} dm \quad (VII)$$

where Q_i , $i=1, \dots, n$ are denoted the generalized forces. Note that

$$\delta W_a = \sum_{i=1}^n Q_i \delta q_i \quad (4.13)$$

change of coordinates $q_i \rightarrow q_i^*$

$$q_i = q_i(t, q_1^*, \dots, q_n^*) \quad (4.14)$$

$$\det \left[\frac{\partial q_i}{\partial q_j^*} \right] \neq 0$$

Define

$$T^* = T^*(t, q^*, \dot{q}^*) = T(t, q, \dot{q}) \quad (4.15)$$

then

$$Q_i^* = \frac{d}{dt} \left(\frac{\partial T^*}{\partial \dot{q}_i^*} \right) - \frac{\partial T^*}{\partial q_i^*} \quad (4.16)$$

where

$$Q_i^* = \sum_{j=1}^n Q_j \frac{\partial q_j}{\partial q_i^*}$$

If we introduce "the variational derivative",

$$\frac{\delta T}{\delta q} = \frac{\partial T}{\partial q} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}} \right) \quad (4.17)$$

we have Lagrange's equations

$$Q + \frac{\delta T}{\delta q} = 0 \quad (4.18)$$

A change of coordinates $q \rightarrow q^*$

$$\frac{\delta T^*}{\delta q^*} = \frac{\delta T}{\delta q} \frac{\partial q}{\partial q^*} \quad (4.19)$$

$$Q^* = Q \frac{\partial q}{\partial q^*}$$

Then

$$Q^* + \frac{\delta T^*}{\delta q^*} = \left(Q + \frac{\delta T}{\delta q} \right) \frac{\partial q}{\partial q^*} \quad (4.20)$$

where $\det \left[\frac{\partial q}{\partial q^*} \right] \neq 0$

Hamilton's principle (optional)

Given a Lagrangian

$$L = L(t, q, \dot{q})$$

and a motion

$$q = q(t), \quad t_1 \leq t \leq t_2$$

The action of the mechanical system in this motion (and during the time interval $[t_1, t_2]$) is defined by

$$A = \int_{t_1}^{t_2} L(t, q(t), \dot{q}(t)) dt \quad (4.21)$$

We now look at a family of motions

$$q_\varepsilon = q_\varepsilon(t) = q(t) + \varepsilon \delta q(t)$$

where ε is a parameter and $\delta q = \delta q(t)$ an arbitrary time function. The corresponding action

$$A = A(\varepsilon) = \int_{t_1}^{t_2} L(t, q_\varepsilon(t), \dot{q}_\varepsilon(t)) dt$$

The variation of A is defined by

$$\delta A = \delta A(q; \delta q) = \frac{dA}{d\varepsilon}(0) \quad (4.22)$$

By derivation under the integral and a partial integration we get

$$\delta A = \delta A(q; \delta q) = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = \int_{t_1}^{t_2} \frac{\partial L}{\partial q} \delta q dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} \quad (4.23)$$

where the variational derivative:

$$\frac{\delta L}{\delta q} = \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right)$$

Note that δA is linear in δq , i.e.

$$\delta A(q; \lambda \delta q) = \lambda \delta A(q; \delta q), \quad \lambda \in \mathbb{R}$$

$$\delta A(q; \delta q + \delta r) = \delta A(q; \delta q) + \delta A(q; \delta r)$$

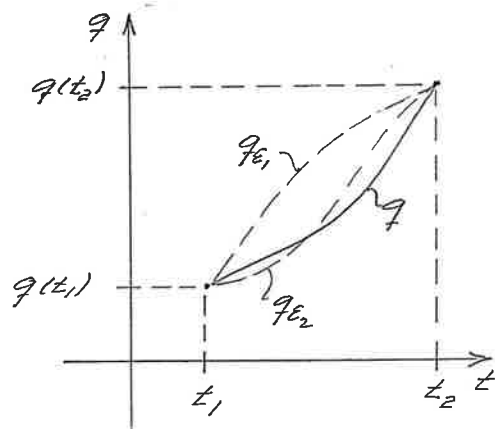
We now assume that

$$q_\varepsilon(t_1) = q(t_1), \quad q_\varepsilon(t_2) = q(t_2), \quad \forall \varepsilon$$

which is equivalent to

$$\delta q(t_1) = \delta q(t_2) = 0 \quad (4.24)$$

This means that all motions in the ε -family are starting in the same configuration $q(t_1)$ and end at $q(t_2)$.



An extremal of A is a motion $q = q(t)$ for which

$$\delta A(q; \delta q) = 0, \quad \forall \delta q = \delta q(t)$$

We may now state Hamilton's principle:

$q = q(t)$ is an extremal for the action A on the space of all motion satisfying

$$q(t_1) = q^1, \quad q(t_2) = q^2$$

(q^1, q^2 fixed configurations) if and only if

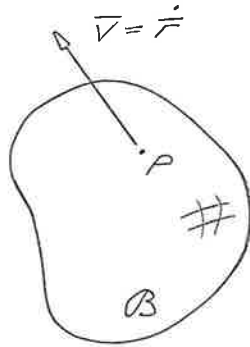
$$\frac{\delta L}{\delta q} = 0 \quad \text{along } q = q(t)$$

5. Structure of the kinetic energy

The kinetic energy

$$E_k = \frac{1}{2} \int_B \mathbf{v} \cdot \mathbf{v} \, dm, \quad E_k \geq 0$$

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial t} + \sum_{i=1}^n \frac{\partial \mathbf{r}}{\partial q_i} \dot{q}_i$$



$$E_k = T(t, q, \dot{q}) =$$

$$\frac{1}{2} \int_B \left| \frac{\partial \mathbf{r}}{\partial t} + \sum_{i=1}^n \frac{\partial \mathbf{r}}{\partial q_i} \dot{q}_i \right|^2 dm = \quad (5.1)$$

$$T_0 + T_1 + T_2 \quad (\text{"the three parts"})$$

$$T_0 = \frac{1}{2} m_{00}, \quad m_{00} = \int_B \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial t} dm$$

$$T_1 = \frac{1}{2} \sum_{i=1}^n (m_{0i} + m_{i0}) \dot{q}_i, \quad m_{0i} = \int_B \frac{\partial \mathbf{r}}{\partial t} \cdot \frac{\partial \mathbf{r}}{\partial q_i} dm = m_{i0}$$

$$T_2 = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n m_{ij} \dot{q}_i \dot{q}_j, \quad m_{ij} = \int_B \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} dm = m_{ji}$$

Scleronomic coordinate system $\Rightarrow$

$$T = T_2$$

With

$$\delta \mathbf{r} = \sum_{i=1}^n \frac{\partial \mathbf{r}}{\partial q_i} \delta q_i$$

one obtains

$$0 \leq \int_B \delta \mathbf{r} \cdot \delta \mathbf{r} \, dm = \sum_{i,j=1}^n a_{ij} \delta q_i \delta q_j$$

and we conclude that

$$T_2 = \frac{1}{2} \sum_{i,j=1}^n m_{ij} \dot{q}_i \dot{q}_j$$

is a positive definite quadratic form in $\dot{q}_i$. We have the following equivalent statements:

- q is a regular coordinate system
- T_2 is a positive definite quadratic form.
- The matrix $[m_{ij}]$ is positive definite

$$T = T_0 + T_1 + T_2$$

into "L's eq." (Lagrange's equations)

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial T_1}{\partial \dot{q}_i} + \frac{\partial T_2}{\partial \dot{q}_i} \right) -$$

$$\frac{\partial}{\partial q_i} (T_0 + T_1 + T_2) = \frac{\partial}{\partial t} \left(\frac{\partial T_1}{\partial \dot{q}_i} \right) + \sum_{j=1}^n \frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} \dot{q}_j +$$

$$\frac{d}{dt} \left(\frac{\partial T_2}{\partial \dot{q}_i} \right) - \frac{\partial}{\partial q_i} (T_0 + T_1 + T_2)$$

and then

$$\left\| \frac{\delta T}{\delta q_i} = \frac{\delta T_2}{\delta q_i} - K_i^t - K_i^g \right. \quad (15.2)$$

where

$$\left\| K_i^t = \frac{\partial}{\partial t} \left(\frac{\partial T_1}{\partial \dot{q}_i} \right) - \frac{\partial T_0}{\partial q_i} \right. \quad \begin{array}{l} \text{transport} \\ \text{inertia force} \end{array} \quad (15.3)$$

$$\left\| K_i^g = \sum_{j=1}^n \frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} \dot{q}_j - \frac{\partial T_1}{\partial q_i} \right. \quad \begin{array}{l} \text{gyroscopic} \\ \text{inertia force} \end{array}$$

Note that

$$\dot{q}_i = 0 \Rightarrow \frac{\delta T}{\delta q_i} = K_i^t$$

$$q\text{-system scleronomic} \Rightarrow \frac{\delta T}{\delta q_i} = \frac{\delta T_2}{\delta q_i}$$

For the gyroscopic inertia force we have

$$\left\| K_i^g = \sum_{j=1}^n \tilde{g}_{ij} \dot{q}_j \right. \quad (15.4)$$

where

$$\left\| \tilde{g}_{ij} = \frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} - \frac{\partial^2 T_1}{\partial q_i \partial \dot{q}_j} = -\tilde{g}_{ji} \right. \quad (15.5)$$

The coefficient matrix $[\tilde{g}_{ij}]$ is thus antisymmetric:

$$[\tilde{g}_{ij}]^T = -[\tilde{g}_{ij}]$$

In Geradin & Rixen (p. 19) the symbol F_i is used for the gyroscopic or complementary inertia force.

The kinetic energy for a rigid body

The kinetic energy

$$E_k = \frac{1}{2} m \bar{V}_C^2 + \frac{1}{2} \bar{\omega} \cdot \underline{I}_C \bar{\omega} \quad (5.6)$$

$\bar{V}_C$: velocity of centre of mass

$\bar{\omega}$: angular velocity

m : mass

$\underline{I}_C$: inertia tensor (with respect to C)

Let x, y, z denote cartesian coordinates of the centre of mass and θ, ψ, ϕ the "Euler angles" (see

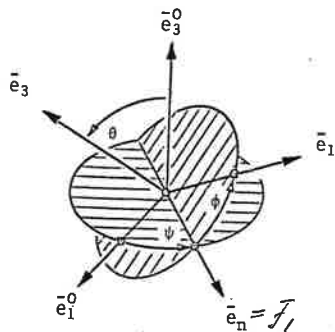
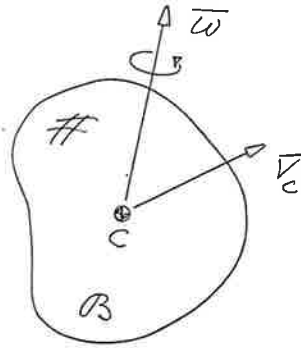
Meriam & Kraige, Dynamics)

Generalized coordinates:

$$q_1 = x, q_2 = y, q_3 = z$$

$$q_4 = \psi, q_5 = \theta, q_6 = \phi$$

$$\bar{\omega} = \bar{e}_3^0 \dot{\psi} + \bar{e}_n \dot{\theta} + \bar{e}_3 \dot{\phi}$$



$$\bar{e}_2^0 = \bar{e}_3^0 \times \bar{e}_1^0$$

$(\bar{e}_1^0, \bar{e}_2^0, \bar{e}_3^0)$ "space-fixed" and

$(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ "body-fixed," $\bar{e}_n = \bar{e}_3 \times \bar{e}_3^0$

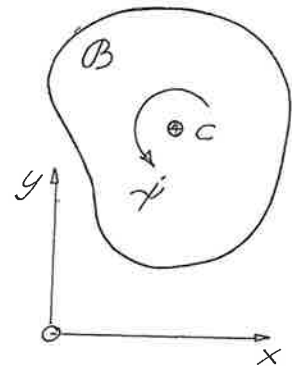
Then

$$E_k = T(\psi, \theta, \phi, \dot{x}, \dot{y}, \dot{z}, \dot{\psi}, \dot{\theta}, \dot{\phi}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{1}{2} (I_1 \sin^2 \theta + I_3 \cos^2 \theta) \dot{\psi}^2 + I_1 \dot{\theta}^2 + I_3 \dot{\phi}^2 + 2 \cos \theta \dot{\psi} \dot{\phi} \quad (5.7)$$

if $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ are principal inertia axis with moments of inertia I_1, I_2 and I_3 respectively.

In plane motion ($\dot{z}=0, \theta=\phi=0$):

$$E_k = T(\dot{x}, \dot{y}, \dot{\psi}) = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) m + \frac{1}{2} I_3 \dot{\psi}^2 \quad (5.8)$$



6. the conservative generalized forces.

$$Q_i = \int_B \bar{F}_a \cdot \frac{\partial \bar{r}}{\partial q_i} dm, \quad i = 1, \dots, n \quad (6.1)$$

where

$$\bar{F}_a = \bar{F}_a^e + \bar{F}_a^i \quad (6.2)$$

The accelerating force $\bar{F}_a$ is said to be conservative if there is a (scalar) potential:

$$V = V(t, \bar{r}; P) \quad (6.3)$$

such that

$$\bar{F}_a = - \frac{\partial V}{\partial \bar{r}} \quad (6.4)$$

Then, by using the "chain-rule":

$$\begin{aligned} Q_i &= \int_B - \frac{\partial V}{\partial \bar{r}} \cdot \frac{\partial \bar{r}}{\partial q_i} dm = - \int_B \frac{\partial V}{\partial q_i} dm = \\ &= - \frac{d}{dq_i} \int_B V dm = - \frac{\partial V}{\partial q_i} \end{aligned} \quad (6.5)$$

$$V = V(t, q) = \int_B V dm$$

V is the potential energy of the system (body).

Lagrange's eq. $\Rightarrow$

$$Q + \frac{\delta T}{\delta q} = - \frac{\partial V}{\partial q} + \frac{\delta T}{\delta q} = 0$$

But

$$\frac{\partial V}{\partial q} = \frac{\delta V}{\delta q}$$

since V does not depend on $\dot{q}$ and consequently:

$$\frac{\partial V}{\partial \dot{q}} = 0$$

Thus L 's eq. may be expressed:

$$\frac{\delta L}{\delta q} = 0 \quad (6.6)$$

where $L = L(t, q, \dot{q})$ is the Lagrangian function defined by

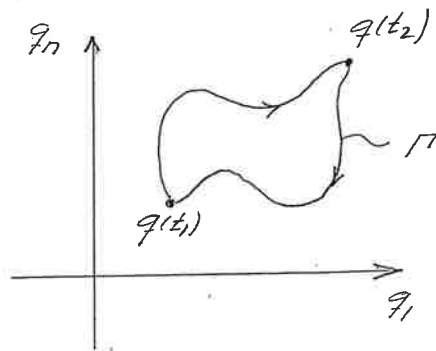
$$L = T - V \quad (6.7)$$

The work done by the conservative forces (in time interval $[t_1, t_2]$) may be written:

$$\int_{t_1}^{t_2} \sum_{i=1}^n Q_i \dot{q}_i dt = - \int_{t_1}^{t_2} \sum_{i=1}^n \frac{\partial V}{\partial q_i} \dot{q}_i dt =$$

$$- \int_{t_1}^{t_2} \frac{dV}{dt} dt = V(q(t_1)) - V(q(t_2))$$

The work is thus independent of the motion between $q(t_1)$ and $q(t_2)$



We may write (Γ closed curve in q -space)

$$\oint_{\Gamma} \sum_{i=1}^n Q_i dq_i = 0$$

Ex. 10 Gravitational potential

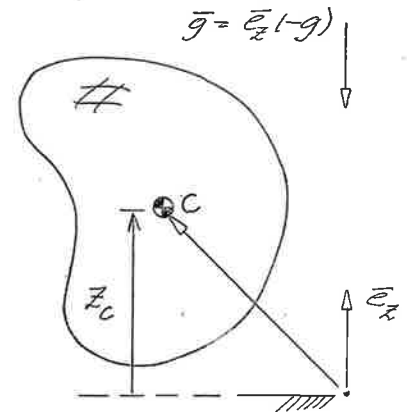
$$v(\vec{r}) = - \vec{g} \cdot \vec{r} =$$

$$g \vec{e}_z \cdot \vec{r} = gz$$

$$V = \int_{\mathcal{B}} v dm =$$

$$g \int_{\mathcal{B}} z dm = mg z_c$$

$$Q_i = - \frac{\partial V}{\partial q_i} = - mg \frac{\partial z_c}{\partial q_i}, \quad z_c = \frac{1}{m} \int_{\mathcal{B}} z dm$$

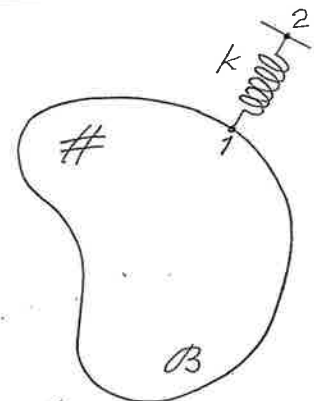


Ex. 11 Elastic potential

$$V = \frac{k}{2} |\vec{r}_1 - \vec{r}_2|^2$$

$$Q_i = - k \left(\frac{\partial \vec{r}_1}{\partial q_i} - \frac{\partial \vec{r}_2}{\partial q_i} \right) \cdot (\vec{r}_1 - \vec{r}_2)$$

Compare with Ex 6!



Ex. 12 Simple planar pendulum.
Revisited.

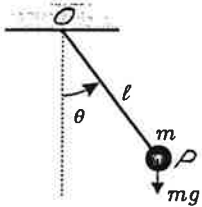


Figure 1.3.1
The non-linear pendulum

Generalized coordinate

$$q = \theta, \quad \dot{q} = \dot{\theta}$$

$$\vec{r}_p = \vec{e}_x l \sin \theta + \vec{e}_z (-l \cos \theta)$$

$$\dot{\vec{r}}_p = \{\vec{e}_x l \cos \theta + \vec{e}_z (l \sin \theta)\} \dot{\theta}$$

$$T = \frac{1}{2} m \dot{\vec{r}}^2 = \frac{1}{2} m l^2 \dot{\theta}^2 = T(\dot{\theta}), \quad T_0 = T_1 = 0$$

Note that:

$$a_{00} = a_{01} = a_{10} = 0, \quad a_{11} = m l^2$$

$$V = m g z_p = -m g l \cos \theta$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l \cos \theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta}, \quad \frac{\partial L}{\partial \theta} = -m g l \sin \theta$$

L's eq.

$$\frac{\delta L}{\delta \theta} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \Rightarrow$$

$$m l^2 \ddot{\theta} + m g l \sin \theta = 0$$

#

A mechanical system is said to be conservative if there is a scalar potential (potential energy).

$$V = V(t, q)$$

such that

$$Q_i = - \frac{\partial V}{\partial q_i}, \quad i = 1, \dots, n \quad (6.8)$$

The generalized forces are then said to be conservative.

The mechanical energy of the system is defined by

$$E = T + V \quad (6.9)$$

A scleronomic system $\Rightarrow$

$$T = T_2 \Rightarrow \quad ("Euler's theorem")^*$$

$$T = \frac{1}{2} \sum_{i=1}^n \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} \Rightarrow$$

$$2\dot{T} = \sum_{i=1}^n \ddot{q}_i \frac{\partial T}{\partial \dot{q}_i} + \sum_{i=1}^n \dot{q}_i \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right)$$

* on homogeneous functions

On the other hand ($T = T_2 = T_2(q, \dot{q})$)

$$\dot{T} = \sum_{i=1}^n \left(\frac{\partial T}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial T}{\partial q_i} \dot{q}_i \right)$$

subtracting $\Rightarrow$

$$\dot{T} = \sum_{i=1}^n -\frac{\delta T}{\delta q_i} \dot{q}_i$$

Now

$$\dot{V} = \sum_{i=1}^n \frac{\partial V}{\partial \dot{q}_i} \ddot{q}_i = \sum_{i=1}^n \frac{\delta V}{\delta \dot{q}_i} \ddot{q}_i$$

$\Rightarrow$

$$\dot{E} = \dot{T} + \dot{V} = - \sum_{i=1}^n \frac{\delta L}{\delta q_i} \dot{q}_i$$

But L 's eq.

$$\frac{\delta L}{\delta q_i} = 0, \quad i = 1, \dots, n$$

consequently:

$$\dot{E} = 0 \quad \Leftrightarrow \quad E = \text{constant} \quad (6.10)$$

7. Non-conservative generalized forces

In order for Q to be conservative it is necessary that

$$\frac{\partial Q_i}{\partial \dot{q}_j} = \frac{\partial Q_j}{\partial \dot{q}_i}, \quad i, j = 1, \dots, n \quad (7.1)$$

If this condition is not fulfilled then Q is non-conservative (dissipative).

An example of a nonconservative force is given by

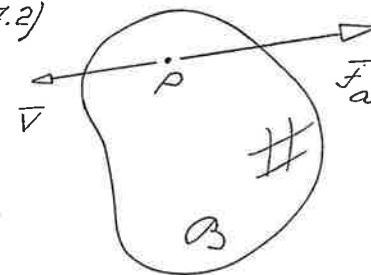
$$\vec{F}_a = -d(t, V; P) \frac{\vec{V}}{V} \quad (7.2)$$

where $V = |\vec{V}| \neq 0$

$$d = d(t, V; P) = d(V)$$

is a given function describing the velocity dependence of the force.

$$Q_i = \int_{\mathcal{B}} \vec{F}_a \cdot \frac{\partial \vec{r}}{\partial q_i} dm = - \int_{\mathcal{B}} d(V) \frac{\vec{V}}{V} \cdot \frac{\partial \vec{r}}{\partial q_i} dm$$



$$\left| \frac{\partial \bar{F}}{\partial \dot{q}_i} = \frac{\partial \bar{V}}{\partial \dot{q}_i} = \bar{p} \right|$$

$$Q_i = - \int_B d(V) \frac{\bar{V}}{V} \cdot \frac{\partial \bar{V}}{\partial \dot{q}_i} dm = - \int_B d(V) \frac{\partial V}{\partial \dot{q}_i} dm$$

$$\text{since } \bar{V} \cdot \frac{\partial \bar{V}}{\partial \dot{q}_i} = V \frac{\partial V}{\partial \dot{q}_i} \quad (\text{use } V^2 = \bar{V} \cdot \bar{V})$$

We may now introduce the dissipation function

$$\left| D = D(t, q, \dot{q}) = \int_B \int_0^V d(\gamma) d\gamma dm \quad (7.3) \right|$$

where

$$V = V(t, q, \dot{q}) = \left| \frac{\partial \bar{F}}{\partial t} + \sum_{i=1}^n \frac{\partial \bar{F}}{\partial \dot{q}_i} \dot{q}_i \right|$$

Then

$$\left| Q_i = - \frac{\partial D}{\partial \dot{q}_i} \quad (7.4) \right|$$

If D is homogeneous of degree m as a function of $\dot{q}$, i.e.

$$\left| D(t, q, \lambda \dot{q}) = \lambda^m D(t, q, \dot{q}) \right|$$

then

$$\left| \sum_{i=1}^n Q_i \dot{q}_i = \sum_{i=1}^n \frac{\partial D}{\partial \dot{q}_i} \dot{q}_i = m D \right|$$

For the function:

$$\left| d = d(t, V; P), \quad V \geq 0 \right|$$

it is reasonable to assume the following properties:

$$\left| d(t, V; P) \geq 0 \right|$$

$$\left| d(t, 0; P) = 0 \right|$$

This implies

$$\left| D(t, q, \dot{q}) \geq 0 \right|$$

(7.5)

$$\left| Q_i(t, q, \dot{q}=0) = 0 \right|$$

Theorem on homogeneous functions:

$$f = f(q_1, \dots, q_n) = f(q).$$

$$f(\lambda q) = \lambda^m f(q) \Rightarrow \sum_{i=1}^n q_i \frac{\partial f}{\partial q_i} = m f$$

The generalized forces for a rigid body

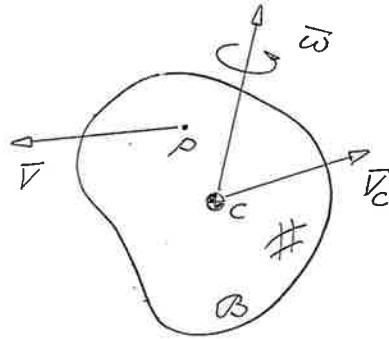
The velocity field for a rigid body:

$$\vec{v} = \vec{v}_c + \vec{\omega} \times (\vec{r} - \vec{r}_c) \quad (7.6)$$

$$\vec{r} = \vec{r}(q; P)$$

$$\vec{\omega} = \vec{\omega}(q, \dot{q})$$

$$\vec{v}_c = \vec{v}_c(q, \dot{q})$$



$$(q = q_1, \dots, q_n; \dot{q} = \dot{q}_1, \dots, \dot{q}_n)$$

The generalized force:

$$Q_i = \int_B \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} dm$$

But

$$\frac{\partial \vec{r}}{\partial q_i} = \vec{v}_{c;i} + \vec{\omega}_{;i} \times (\vec{r} - \vec{r}_c) \quad i = 1, \dots, n$$

where $\vec{v}_{c;i}$ and $\vec{\omega}_{;i}$ are vectors defined analogous to $\vec{v}_c$ and $\vec{\omega}$ when t is changed for q . $\vec{v}_{c;i}$ and $\vec{\omega}_{;i}$ are independent of the particle P .

$$\vec{v} = \sum_{i=1}^n \frac{\partial \vec{r}}{\partial q_i} \dot{q}_i = \sum_{i=1}^n \vec{v}_{c;i} \dot{q}_i + \left(\sum_{i=1}^n \vec{\omega}_{;i} \dot{q}_i \right) \times (\vec{r} - \vec{r}_c)$$

and then by comparison with (7.6) we get:

$$\vec{v}_c = \sum_{i=1}^n \vec{v}_{c;i} \dot{q}_i, \quad \vec{\omega} = \sum_{i=1}^n \vec{\omega}_{;i} \dot{q}_i$$

and consequently:

$$\vec{v}_{c;i} = \frac{\partial \vec{v}_c}{\partial \dot{q}_i}, \quad \vec{\omega}_{;i} = \frac{\partial \vec{\omega}}{\partial \dot{q}_i} \quad (7.7)$$

and

$$Q_i = \int_B \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} dm = \vec{v}_{c;i} \cdot \int_B \vec{F} dm +$$

$$\vec{\omega}_{;i} \cdot \int_B (\vec{r} - \vec{r}_c) \times \vec{F} dm =$$

$$\vec{v}_{c;i} \cdot \vec{F}^e + \vec{\omega}_{;i} \cdot \vec{M}_c^e \quad (7.8)$$

where $\vec{F}^e$ is the sum of the external forces on B and $\vec{M}_c^e$ their moment sum.

LECTURE 3

EQUATIONS OF MOTION

Dissipation
Energy balance
Stability

8. Lagranges equations in
the general case

$$T = T_0 + T_1 + T_2 \quad \text{Kinetic energy}$$

$$V \quad \text{Potential energy}$$

$$L = T - V \quad \text{Lagrangian function}$$

$$L_2 = T_2 - V \quad \text{Relative Lagrangian}$$

$$K_i^t \quad \text{Transport inertia force}$$

$$K_i^g \quad \text{Gyroscopic inertia force}$$

$$D \quad \text{Dissipation function}$$

$$Q_i^{ext} \quad \text{Non-cons. external force}$$

$$Q_i^{ext} - \frac{\partial D}{\partial \dot{q}_i} + \frac{\delta L}{\delta q_i} = 0 \quad (18.1)$$

or

$$Q_i^{ext} - \frac{\partial D}{\partial \dot{q}_i} + K_i^t + K_i^g + \frac{\delta L_2}{\delta q_i} = 0 \quad (18.2)$$

$$i = 1, \dots, n.$$

An explicit calculation gives:

$$\frac{\delta T_2}{\delta q_i} = - \sum_{j=1}^n m_{ij} \ddot{q}_j - \sum_{j=1}^n \sum_{k=1}^n C_{ijk} \dot{q}_j \dot{q}_k - \sum_{j=1}^n \frac{\partial m_{ij}}{\partial t} \dot{q}_j \quad (8.3)$$

where $(m_{ij} = m_{ji} = \int_B \frac{\partial \bar{r}}{\partial q_i} \cdot \frac{\partial \bar{r}}{\partial q_j} dm)$

$$C_{ijk} = \frac{1}{2} \left(\frac{\partial m_{ij}}{\partial q_k} - \frac{\partial m_{jk}}{\partial q_i} + \frac{\partial m_{ki}}{\partial q_j} \right)$$

Furthermore:

$$\begin{aligned} K_i^t &= \frac{\partial m_{oi}}{\partial t} - \frac{1}{2} \frac{\partial m_{oo}}{\partial q_i} \\ K_i^g &= \sum_{j=1}^n \left(\frac{\partial m_{oi}}{\partial q_j} - \frac{\partial m_{oj}}{\partial q_i} \right) \dot{q}_j \end{aligned} \quad (8.4)$$

We may identify (cf. eq. (5.5)!))

$$\tilde{g}_{ij} = \frac{\partial m_{oi}}{\partial q_j} - \frac{\partial m_{oj}}{\partial q_i} = -\tilde{g}_{ji}$$

L's eq. then result in the following equations of motion

$$\begin{aligned} & \sum_{j=1}^n m_{ij} \ddot{q}_j + \sum_{j=1}^n \sum_{k=1}^n C_{ijk} \dot{q}_j \dot{q}_k + \sum_{j=1}^n \frac{\partial m_{ij}}{\partial t} \dot{q}_j + \\ & \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = -K_i^t - K_i^g + Q_i^{ext}, \quad i=1, \dots, n \end{aligned} \quad (VIII)$$

We look for a solution

$$q_i = q_i(t), \quad i=1, \dots, n$$

satisfying the initial conditions

$$\begin{aligned} q_i(0) &= q_i^0 \\ \dot{q}_i(0) &= \dot{q}_i^0 \end{aligned} \quad (8.5)$$

where

$$q_i^0, \dot{q}_i^0 \quad i=1, \dots, n$$

are given constants.

The system is called non-disturbed if

$$Q_i^{ext}(t) \equiv 0, \quad i=1, \dots, n$$

9. The energy balance

We introduce the generalized
(Lagrangian) mechanical energy

$$E = \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \quad (9.1)$$

Not that for a scleronomic system

$$2T = 2T_2 = \sum_{i=1}^n \frac{\partial T_2}{\partial \dot{q}_i} \dot{q}_i = \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i$$

and then

$$E = 2T - L = 2T - (T - V) = T + V$$

i.e. the usual mechanical energy

The generalized power^{*} is defined by

$$P = \sum_{i=1}^n Q_i \dot{q}_i \quad (Q_i + \frac{\delta L}{\delta \dot{q}_i} = 0) \quad (9.2)$$

Then we have the energy balance theorem

$$\dot{E} = P - \frac{\partial L}{\partial t} \quad (IX)$$

* The power expended by Q .

With

$$Q_i = Q_i^{\text{ext}} - \frac{\partial D}{\partial \dot{q}_i}$$

We get

$$P = \sum_{i=1}^n Q_i^{\text{ext}} \dot{q}_i - \sum_{i=1}^n \frac{\partial D}{\partial \dot{q}_i} \dot{q}_i$$

If D homogeneous of degree m in $\dot{q}$
then

$$P = P^{\text{ext}} - mD$$

(9.3)

$$P^{\text{ext}} = \sum_{i=1}^n Q_i^{\text{ext}} \dot{q}_i$$

and the energy balance is given by

$$\dot{E} = P^{\text{ext}} - mD - \frac{\partial L}{\partial t} \quad (9.4)$$

If $P^{\text{ext}} = 0$, $\frac{\partial L}{\partial t} = 0$ and $D \geq 0$ then

$$\dot{E} = -mD \leq 0 \quad (9.5)$$

E is non-increasing!

The relative mechanical energy is defined by

$$\| E_e = \sum_{i=1}^n \frac{\partial L_2}{\partial \dot{q}_i} \dot{q}_i - L_2 \quad (9.6)$$

By using the fact that the gyroscopic forces are powerless, i.e.

$$\left| \sum_{i=1}^n k_i^g \cdot \dot{q}_i = \sum_{i,j=1}^n \tilde{g}_{ij} \dot{q}_i \dot{q}_j = 0 \right.$$

($\tilde{g}_{ij} = -\tilde{g}_{ji}$) we may show that

$$\| \dot{E}_e = P^{ext} - mD - P^t - \frac{\partial L_2}{\partial t} \quad (9.7)$$

where

$$\left| P^t = \sum_{i=1}^n k_i^t \cdot \dot{q}_i \right.$$

is the power of the transport inertia forces and we have assumed a homogeneous dissipation function of degree m .

10. Equilibrium and stability

The equations of motion:

$$\left| \sum_{j=1}^n m_{ij} \ddot{q}_j + \sum_{j=1}^n \sum_{k=1}^n C_{ijk} \dot{q}_j \dot{q}_k + \frac{1}{2} \sum_{j=1}^n \frac{\partial m_{ij}}{\partial t} \dot{q}_j + \right.$$

$$\left. \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = -K_i^t - K_i^g + Q_i^{ext}, \quad i=1, \dots, n \quad (VIII)$$

Equilibrium: A configuration q^0 is an equilibrium configuration to the nondisturbed system if:

$$\| q_i(t) \equiv q_i^0, \quad Q_i^{ext} \equiv 0 \quad (10.1)$$

is a solution to the equations of motion. If we insert:

$$\left| q_i(t) = q_i^0, \quad \dot{q}_i(t) = 0, \quad \ddot{q}_i(t) = 0 \right.$$

$$\left| Q_i^{ext}(t) = 0 \quad i=1, \dots, n \right.$$

into (VIII) we get:

$$\left| \frac{\partial V}{\partial q_i} = - \frac{d}{dt} \left(\frac{\partial T_1}{\partial \dot{q}_i} \right) + \frac{\partial T_0}{\partial q_i} \quad (10.2) \right.$$

for $q_i = q_i^0$.

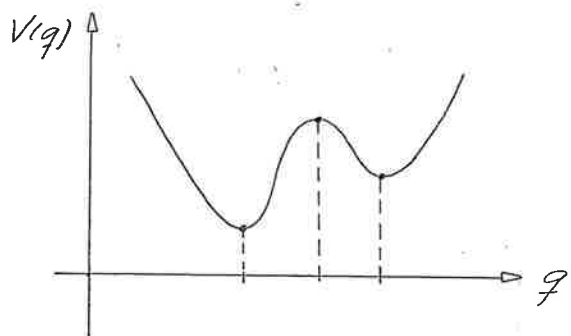
or

$$\left| \frac{d}{dq_i} (V - T_0) = - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) \right. \quad (10.3)$$

Scleronomic system $\Rightarrow T_0 = T_1 \equiv 0 \Rightarrow$

$$\left| \frac{\partial V}{\partial q_i} (q_i^0) = 0 \right.$$

i.e. the equilibrium configurations of the undisturbed system are the configurations for which the potential energy is stationary (max. or min.)



A weaker condition than scleronomy is:

$$\left| \frac{\partial m_{oi}}{\partial t} = 0, \quad i = 1, \dots, n \right.$$

Then from (10.3) it follows

$$\left| \frac{\partial V^*}{\partial q_i} (q_i^0) = 0 \right. \quad (10.4)$$

where

$$\left| V^* = V - T_0 \right. \quad (10.5)$$

is the so-called modified* potential.

Stability: The system configuration coordinates and the corresponding velocity coordinates

$$\left| q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n = q; \dot{q} \right.$$

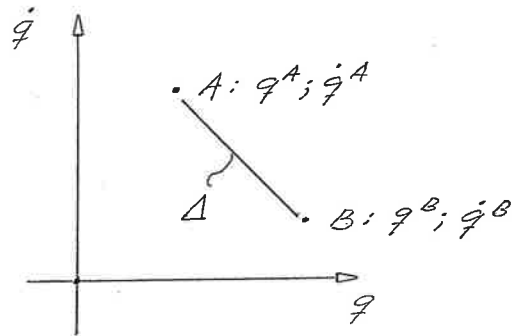
is said to represent the state of the system. The set of all possible states of the system is a region $\mathcal{P} \subseteq \mathbb{R}^{2n}$

We may introduce the following metric (distance) in state space:

$$\left| \Delta[(q^A; \dot{q}^A) - (q^B; \dot{q}^B)] = \right.$$

$$\left| \sqrt{\sum_{i=1}^n ((q_i^A - q_i^B)^2 + (\dot{q}_i^A - \dot{q}_i^B)^2)} \right. \quad (10.6)$$

* efficient



An equilibrium state (with $Q^{ext} \equiv 0$)
 $q = q^0$, $\dot{q} = 0$ is Liapunov stable
 if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that:

$$\Delta[(q(0); \dot{q}(0)) - (q^0; 0)] < \delta \Rightarrow$$

$$\Delta[(q(t); \dot{q}(t)) - (q^0; 0)] < \varepsilon, t > 0$$

We have the following

Dirichlet's stability theorem:

Let q^0 be a configuration at which the potential energy $V = V(q)$ has a local minimum. Then for a scleronomic undisturbed ($Q^{ext} \equiv 0$) system with

no internal dissipation ($D \equiv 0$) the equilibrium state $(q^0; 0)$ is Liapunov stable.

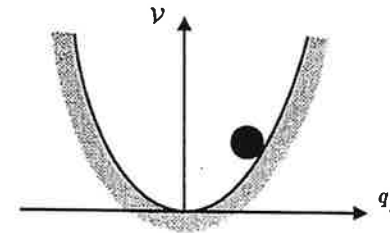


Figure 2.1.1
 Potential of a stable system

An equilibrium state (with $Q^{ext} \equiv 0$)
 $q = q^0$, $\dot{q} = 0$ is asymptotically stable
 if $\exists \delta > 0$ such that:

$$\Delta[(q(0); \dot{q}(0)) - (q^0; 0)] < \delta \Rightarrow$$

$$\lim_{t \rightarrow \infty} \Delta[(q(t); \dot{q}(t)) - (q^0; 0)] = 0$$

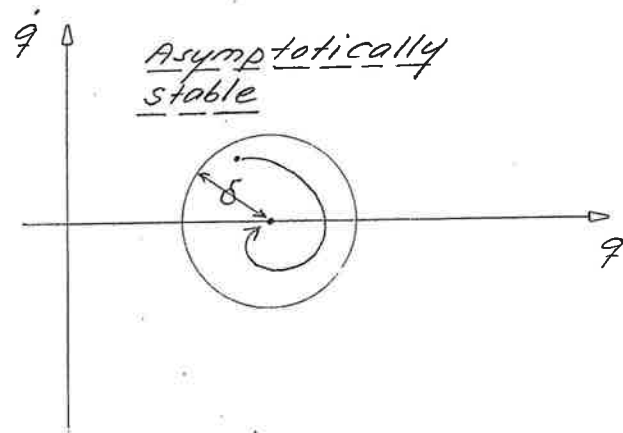
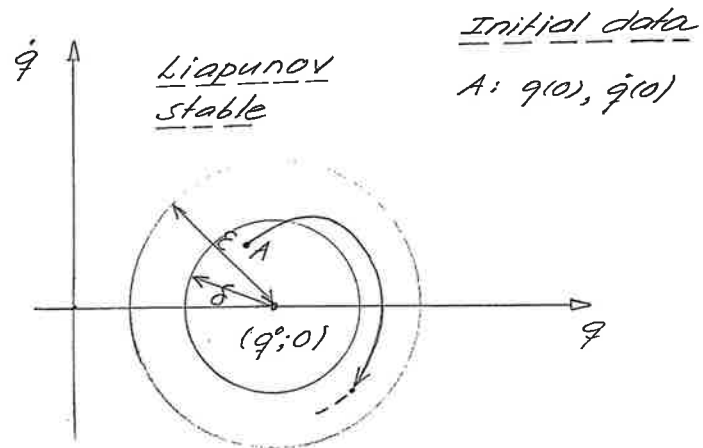
The extended Dirichlet stability theorem

Let q^0 be a configuration at which the potential energy $V = V(q)$ has a

local minimum. Then for a scleronomic undisturbed ($Q^{ext} \equiv 0$) system with dissipation function:

$$D(q, \dot{q}) < 0, \quad \dot{q} \neq 0 \quad (10.7)$$

the equilibrium state is asymptotically stable.



The proof of these theorems are based on the energy balance:

$$\dot{E} = -mD \leq 0 \quad (10.8)$$

$$E = T + V$$

#

LECTURE 4

EQUATIONS OF MOTION

Linearization
Mechanical vibrations

71

11. Linearization

We start from Lagrange's equations on the form:

$$\left\| \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i^{int} + Q_i^{ext} \quad (11.1) \right.$$

where the Lagrangian function:

$$\left\| L = L(q, \dot{q}) \right.$$

does not depend explicitly on time t .

"The internal force" $Q_i^{int} = Q_i^{int}(q, \dot{q})$ is characterized by:

$$\left\| \begin{aligned} Q_i^{int}(q, 0) &= 0, \text{ (no velocity} \rightarrow \text{no force)} \\ \sum_{i=1}^n Q_i^{int} \cdot \dot{q}_i &\leq 0, \text{ ("non-positive power")} \end{aligned} \right. \quad (11.2)$$

By comparison with (VIII) we may assume that:

$$\left\| \frac{\partial m_{i0}}{\partial t} = 0 \quad i = 0, \dots, n \right.$$

$$L = T_2 - V^*$$

$$Q_i^{int} = - \frac{\partial D}{\partial \dot{q}_i} + k_i q$$

Now assume that

$$q_i = q_i^0 ; \quad \dot{q} = 0$$

is an equilibrium state. Then

$$\frac{\partial L}{\partial q_i}(q^0, 0) = 0$$

We put

$$q_i = q_i^0 + r_i, \quad \dot{q}_i = \dot{r}_i, \quad \ddot{q}_i = \ddot{r}_i \quad (11.3)$$

i.e. $r_i; \dot{r}_i$ is the "deviation" from the equilibrium state. If we put this into the equations of motion (11.1) and formally linearize by considering

$$r_i, \dot{r}_i, \ddot{r}_i$$

small quantities to the first order, we get the linearized equations of motion:

$$\sum_{j=1}^n (m_{ij} \ddot{r}_j + b_{ij} \dot{r}_j + k_{ij} r_j) = Q_i^{ext}$$

where

$$m_{ij} = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right)_0 \quad (IX)$$

$$b_{ij} = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right)_0 - \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j} \right)_0 - \left(\frac{\partial Q_i^{int}}{\partial \dot{q}_j} \right)_0$$

$$k_{ij} = - \left(\frac{\partial^2 L}{\partial q_i \partial q_j} \right)_0$$

(The notation $()_0$ means that, after the derivation is executed, we put $(q, \dot{q}) = (q_0, 0)$).

The coefficients m_{ij} and k_{ij} fulfill

$$\begin{aligned} m_{ij} &= m_{ji} \\ k_{ij} &= k_{ji} \end{aligned} \quad (\text{symmetry}) \quad (11.4)$$

We have

$$Q_i^{int}(q, \dot{q}) = Q_i^{int}(q^0 + r, \dot{r}) =$$

$$= \sum_{j=1}^n \left(\frac{\partial Q_i^{int}}{\partial \dot{q}_j} \right)_0 \dot{r}_j + \sum_{j=1}^n \left(\frac{\partial Q_i^{int}}{\partial \dot{q}_i} \right)_0 \dot{r}_j + \dots + \text{higher}$$

order terms (in r_j and $\dot{r}_j$).

$$Q_i^{int}(q, 0) = 0 \Rightarrow \frac{\partial Q_i^{int}}{\partial \dot{q}_i}(q, 0) = 0$$

Thus

$$0 \geq \sum_{i=1}^n Q_i^{int} \dot{q}_i = \sum_{i,j=1}^n \left(\frac{\partial Q_i^{int}}{\partial \dot{q}_j} \right)_0 \dot{r}_i \dot{r}_j + \dots$$

But

$$\sum_{i,j=1}^n \left(\frac{\partial Q_i^{int}}{\partial \dot{q}_j} \right)_0 \dot{r}_i \dot{r}_j = \sum_{i,j=1}^n \left\{ \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right)_0 - \right.$$

$$\left. \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right)_0 + b_{ij} \right\} \dot{r}_i \dot{r}_j = \sum_{i,j=1}^n c_{ij} \dot{r}_i \dot{r}_j$$

where

$$c_{ij} = \frac{1}{2} (b_{ij} + b_{ji}) = c_{ji}$$

and we may then assume the following property of the coefficients c_{ij}

$$\sum_{i,j=1}^n c_{ij} \dot{r}_i \dot{r}_j \geq 0, \quad \forall \dot{r}_i \quad (11.5)$$

which means that the symmetric matrix $[c_{ij}]$ is positive semidefinite

We call

$$\underline{C} = [c_{ij}]$$

the damping matrix. We also introduce the gyroscopic matrix:

$$\underline{G} = [g_{ij}] \quad (11.6)$$

$$g_{ij} = \frac{1}{2} (b_{ij} - b_{ji}) = -g_{ji}$$

and we have the relation:

$$\underline{B} = [b_{ij}] = \underline{C} + \underline{G}$$

Note that:

$$\underline{C}^T = \underline{C}, \quad \bar{a}^T \underline{C} \bar{a} \geq 0$$

for all column vectors $\bar{a}$. Furthermore

$$\underline{G}^T = -\underline{G}, \quad \bar{a}^T \underline{G} \bar{a} = 0, \quad \forall \bar{a}$$

The symmetric matrix

$$\underline{M} = [m_{ij}] \quad (11.8)$$

is called the mass matrix. We assume that $\underline{M}$ is strictly positive definite

The symmetric matrix

$$\underline{K} = [k_{ij}] \quad (11.9)$$

is called the stiffness matrix.

The linearized equations of motion may then be written in matrix form

$$\underline{M} \ddot{\underline{F}} + \underline{B} \dot{\underline{F}} + \underline{K} \underline{F} = \underline{\bar{Q}}^{ext} \quad (X)$$

$$\underline{F} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}, \quad \underline{\bar{Q}}^{ext} = \begin{pmatrix} Q_1^{ext} \\ \vdots \\ Q_n^{ext} \end{pmatrix}$$

This is the equation for

MECHANICAL VIBRATIONS !

Remember linear algebra?

$$\underline{A} \text{ symmetric matrix} \Leftrightarrow \bar{a}^T \underline{A} \bar{b} = \bar{b}^T \underline{A} \bar{a}, \quad \forall \bar{a}, \bar{b}$$

$$\underline{A} \text{ antisymmetric matrix} \Leftrightarrow \bar{a}^T \underline{A} \bar{b} = -\bar{b}^T \underline{A} \bar{a}, \quad \forall \bar{a}, \bar{b}$$

$$\underline{A} \text{ positive definite} \Leftrightarrow \bar{a}^T \underline{A} \bar{a} \geq 0, \quad \forall \bar{a}$$

$$\underline{A} \text{ strictly positive definite} \Leftrightarrow \bar{a}^T \underline{A} \bar{a} > 0, \quad \forall \bar{a} \neq \bar{0}$$

Every $\underline{A}$ may be written $\underline{A} = \underline{B} + \underline{C}$

where $\underline{B}^T = \underline{B}$, $\underline{C}^T = -\underline{C}$ are unique:

$$\underline{B} = \frac{1}{2}(\underline{A} + \underline{A}^T), \quad \underline{C} = \frac{1}{2}(\underline{A} - \underline{A}^T)$$

$\underline{A}$ strictly positive definite $\Rightarrow$

$$\det \underline{A} \neq 0 \Leftrightarrow \underline{A}^{-1} \text{ exists.}$$

We will, from now on, use $\bar{q}$ instead of $\bar{r}$ as the deviation from the equilibrium configuration and $\bar{p}$ as the external generalized force:

$$\underline{M}\ddot{\bar{q}} + \underline{B}\dot{\bar{q}} + \underline{K}\bar{q} = \bar{p} \quad (XI)$$

$$\bar{q} = [q_i] = \begin{pmatrix} q_1 \\ \vdots \\ q_n \end{pmatrix} \in \mathbb{R}^n$$

$$\underline{M} = [M_{ij}] = \begin{pmatrix} m_{11} & \dots & m_{1n} \\ m_{21} & \dots & m_{2n} \\ \vdots & & \vdots \\ m_{n1} & \dots & m_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

$$\underline{B} = [B_{ij}] = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ b_{21} & \dots & b_{2n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

$$\underline{K} = [K_{ij}] = \begin{pmatrix} k_{11} & \dots & k_{1n} \\ k_{21} & \dots & k_{2n} \\ \vdots & & \vdots \\ k_{n1} & \dots & k_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

$$\bar{p} = [p_i] = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} \in \mathbb{R}^n$$

Physical properties:

$$\underline{M}^T = \underline{M} \quad (\text{symmetric})$$

$$\bar{a}^T \underline{M} \bar{a} > 0, \quad \forall \bar{a} \neq \bar{0}$$

$$\underline{B} = \underline{C} + \underline{G}, \quad \underline{C}^T = \underline{C}, \quad \underline{G}^T = -\underline{G} \quad (11.10)$$

$$\bar{a}^T \underline{C} \bar{a} \geq 0, \quad \forall \bar{a}$$

$$\underline{K}^T = \underline{K}$$

$$\bar{a}^T \underline{K} \bar{a} \geq 0, \quad \forall \bar{a}$$

The kinetic energy of the linear system:

$$T_L = T_L(\dot{q}) = \frac{1}{2} \dot{\bar{q}}^T \underline{M} \dot{\bar{q}} = \frac{1}{2} \sum_{i,j=1}^n m_{ij} \dot{q}_i \dot{q}_j \quad (11.11)$$

The potential energy of the linear system:

$$V_L = V_L(q) = \frac{1}{2} \bar{q}^T \underline{K} \bar{q} = \frac{1}{2} \sum_{i,j=1}^n k_{ij} q_i q_j \quad (11.12)$$

The gyroscopic potential of the linear system:

$$G_L(q, \dot{q}) = \frac{1}{2} \bar{q}^T \underline{G} \dot{\bar{q}} = \frac{1}{2} \sum_{i,j=1}^n g_{ij} q_i \dot{q}_j \quad (11.13)$$

Note that: $G_L(q, \dot{q}) = -\frac{1}{2} \dot{\bar{q}}^T \underline{G} \bar{q}$

The (Rayleigh) dissipation function of the linear system:

$$D_L(\dot{q}) = \frac{1}{2} \dot{\bar{q}}^T \underline{C} \dot{\bar{q}} = \frac{1}{2} \sum_{i,j=1}^n c_{ij} \dot{q}_i \dot{q}_j \quad (11.14)$$

The Lagrangian of the linear system:

$$L_L(q, \dot{q}) = T_L - V_L - G_L \quad (11.15)$$

Lagrange's equations of the linear system:

$$\frac{d}{dt} \left(\frac{\partial L_L}{\partial \dot{q}_i} \right) - \frac{\partial L_L}{\partial q_i} + \frac{\partial D_L}{\partial \dot{q}_i} = P_i, \quad i=1, \dots, n$$

$\Downarrow$

$$\underline{M} \ddot{\bar{q}} + (\underline{C} + \underline{G}) \dot{\bar{q}} + \underline{K} \bar{q} = \bar{P} \quad (11.16)$$

The mechanical energy of the linear system:

$$E_L = \sum_{i=1}^n \frac{\partial L_L}{\partial \dot{q}_i} \dot{q}_i - L_L = T_L + V_L$$

$$\dot{E}_L = \dot{T}_L + \dot{V}_L = \frac{d}{dt} \frac{1}{2} \dot{\bar{q}}^T \underline{M} \dot{\bar{q}} + \frac{d}{dt} \frac{1}{2} \bar{q}^T \underline{K} \bar{q} =$$

$$\dot{\bar{q}}^T \underline{M} \ddot{\bar{q}} + \dot{\bar{q}}^T \underline{K} \bar{q} = \dot{\bar{q}}^T (\underline{M} \ddot{\bar{q}} + \underline{K} \bar{q}) =$$

$$- \dot{\bar{q}}^T \underline{B} \dot{\bar{q}} = \dot{\bar{q}}^T \bar{P} - \dot{\bar{q}}^T \underline{C} \dot{\bar{q}}$$

Where we have used (II) and that

$$\dot{\bar{q}}^T \underline{G} \dot{\bar{q}} \equiv 0.$$

Then

$$\dot{\bar{q}}^T \underline{C} \dot{\bar{q}} \geq 0 \Rightarrow \dot{E}_L \leq \dot{\bar{q}}^T \bar{P} \quad (11.17)$$

i.e. the change in mechanical energy is less than or equal to the power of the external forces.

Statics $\Leftrightarrow \ddot{\bar{q}} = \bar{0} \Leftrightarrow K\bar{q} = \bar{p}$

K strictly positive definite $\Leftrightarrow$

$$\bar{q} = K^{-1}\bar{p} \quad (11.18)$$

Change of coordinates: $\bar{q} \rightarrow \bar{q}^*$

$$\bar{q} = \underline{A} \bar{q}^* \quad (\Leftrightarrow \bar{q}^* = \underline{A}^{-1} \bar{q}) \quad (11.19)$$

$\underline{A} \in \mathbb{R}^{n \times n}$ constant, non-singular matrix, i.e.

$$\det \underline{A} \neq 0$$

Then

$$\ddot{\bar{q}} = \underline{A} \ddot{\bar{q}}^*, \quad \dot{\bar{q}} = \underline{A} \dot{\bar{q}}^*$$

$$\underline{M} \underline{A} \ddot{\bar{q}}^* + \underline{B} \underline{A} \dot{\bar{q}}^* + K \underline{A} \bar{q}^* =$$

$$\underline{M} \ddot{\bar{q}} + \underline{B} \dot{\bar{q}} + K \bar{q} = \bar{p}$$

$\Rightarrow$ multiply from the left with $\underline{A}^T$

$$\underline{A}^T \underline{M} \underline{A} \ddot{\bar{q}}^* + \underline{A}^T \underline{B} \underline{A} \dot{\bar{q}}^* + \underline{A}^T K \underline{A} \bar{q}^* = \underline{A}^T \bar{p}$$

$$2T_L = \dot{\bar{q}}^T \underline{M} \dot{\bar{q}} = (\underline{A} \dot{\bar{q}}^*)^T \underline{M} \underline{A} \dot{\bar{q}}^* =$$

$$\dot{\bar{q}}^{*T} \underbrace{\underline{A}^T \underline{M} \underline{A}}_{= \underline{M}^*} \dot{\bar{q}}^* = \dot{\bar{q}}^{*T} \underline{M}^* \dot{\bar{q}}^* = 2T_L^*$$

$$2V_L = \bar{q}^T K \bar{q} = \bar{q}^{*T} K^* \bar{q}^* = 2V_L^*$$

$$K^* = \underline{A}^T K \underline{A}$$

$$2G_L = \bar{q}^T \underline{G} \dot{\bar{q}} = \bar{q}^{*T} \underline{G}^* \dot{\bar{q}}^* = 2G_L^*$$

$$\underline{G}^* = \underline{A}^T \underline{G} \underline{A}$$

$$2D_L = \dot{\bar{q}}^T \underline{C} \dot{\bar{q}} = \dot{\bar{q}}^{*T} \underline{C}^* \dot{\bar{q}}^* = 2D_L^*$$

$$\underline{C}^* = \underline{A}^T \underline{C} \underline{A}$$

Summary: Change of coordinates:

$$\bar{q} = \underline{A} \bar{q}^* \quad \det \underline{A} \neq 0$$

$$\underline{M}^* = \underline{A}^T \underline{M} \underline{A}$$

$$\underline{B}^* = \underline{A}^T \underline{B} \underline{A}$$

$$K^* = \underline{A}^T K \underline{A}$$

(11.20)

and for the external force

$$\| \bar{P}^* = \underline{A}^T \bar{P} \quad (11.21)$$

$$\| \begin{aligned} \underline{M} \ddot{\bar{q}} + \underline{B} \dot{\bar{q}} + \underline{K} \bar{q} &= \bar{P} \\ \Downarrow \\ \underline{M}^* \ddot{\bar{q}}^* + \underline{B}^* \dot{\bar{q}}^* + \underline{K}^* \bar{q}^* &= \bar{P}^* \end{aligned} \quad (11.22)$$

Note that

$$\underline{M}^T = \underline{M} \Rightarrow \underline{M}^{*T} = \underline{M}^*$$

$$\underline{G}^T = -\underline{G} \Rightarrow \underline{G}^{*T} = -\underline{G}^*$$

$$\underline{K} \text{ positive def.} \Rightarrow \underline{K}^* \text{ positive def.}$$

The dynamical equations of a
the vibrating system are given by:

$$\| \underline{M} \ddot{\bar{q}} + \underline{B} \dot{\bar{q}} + \underline{K} \bar{q} = \bar{P} \quad (11.23)$$

We may introduce the mechanical impedance (or "dynamical stiffness")

$$\| \underline{Z}(s) = \underline{M}s^2 + \underline{B}s + \underline{K}, \quad s \in \mathbb{C} \quad (11.24)$$

which is a matrix polynomial in the complex variable s , with components:

$$\| \underline{Z}_{ij}(s) = m_{ij}s^2 + b_{ij}s + k_{ij} \quad (11.25)$$

Then formally, we may write (11.24) on the form $(s \rightarrow \frac{d}{dt})$:

$$\| \underline{Z}(\frac{d}{dt}) \bar{q} = \bar{P} \quad (11.26)$$

$$(\frac{d}{dt})^2 = \frac{d^2}{dt^2}$$

The differential operator:

$$\| \underline{Z}(\frac{d}{dt}) = \underline{M} \frac{d^2}{dt^2} + \underline{B} \frac{d}{dt} + \underline{K} \quad (11.27)$$

is linear:

$$\underline{Z}\left(\frac{d}{dt}\right)(\alpha_1 \bar{q}_1 + \alpha_2 \bar{q}_2) = \alpha_1 \underline{Z}\left(\frac{d}{dt}\right)\bar{q}_1 + \alpha_2 \underline{Z}\left(\frac{d}{dt}\right)\bar{q}_2 \quad (11.28)$$

where $\bar{q}_1 = \bar{q}_1(t)$, $\bar{q}_2 = \bar{q}_2(t)$ and α_1, α_2 are constants.

This has the following consequence:

$$\underline{Z}\left(\frac{d}{dt}\right)\bar{q}_1 = \bar{P}_1, \quad \underline{Z}\left(\frac{d}{dt}\right)\bar{q}_2 = \bar{P}_2 \Rightarrow \quad (11.29)$$

$$\underline{Z}\left(\frac{d}{dt}\right)(\bar{q}_1 + \bar{q}_2) = \bar{P}_1 + \bar{P}_2$$

which is the mathematical expression for the superposition principle.

We introduce the null-space (kernel) to the impedance operator. Definition:

$$\mathcal{N}(\underline{Z}\left(\frac{d}{dt}\right)) = \{ \bar{q} = \bar{q}(t) : \underline{Z}\left(\frac{d}{dt}\right)\bar{q} = \bar{0} \} \quad (11.30)$$

This is a linear space, i.e.

$$\bar{q}_1, \bar{q}_2 \in \mathcal{N}(\underline{Z}\left(\frac{d}{dt}\right)) \Rightarrow \alpha_1 \bar{q}_1 + \alpha_2 \bar{q}_2 \in \mathcal{N}(\underline{Z}\left(\frac{d}{dt}\right)) \quad (11.31)$$

$$\dim \mathcal{N}(\underline{Z}\left(\frac{d}{dt}\right)) = 2n$$

A motion $\bar{q} = \bar{q}(t) \in \mathcal{N}(\underline{Z}\left(\frac{d}{dt}\right))$ is called a free vibration ($\bar{P} \equiv \bar{0}$)

A motion $\bar{q} = \bar{q}(t)$ satisfying:

$$\underline{Z}\left(\frac{d}{dt}\right)\bar{q} = \bar{P}$$

is called a forced vibration.

Let $\bar{q}_p$ be a specific solution to (11.23)

(a particulate solution) then all solutions to (11.23) are given by

$$\bar{q}(t) = \bar{q}_p(t) + \bar{q}_f(t) \quad (11.32)$$

where $\bar{q}_f$ is a free vibration.

Proof (cf. lin. alg.):

$$\underline{Z}\left(\frac{d}{dt}\right)\bar{q} = \bar{P} \Rightarrow \underline{Z}\left(\frac{d}{dt}\right)(\bar{q} - \bar{q}_p) = \bar{0}$$

$$\Rightarrow \bar{q} - \bar{q}_p \in \mathcal{N}(\underline{Z}\left(\frac{d}{dt}\right)) \quad \square$$

The initial-value problem for the forced vibrations is given by:

$$\mathcal{L}\left(\frac{d}{dt}\right)\bar{q} = \bar{P}$$

$$\bar{q}(0) = \bar{q}_0, \quad \dot{\bar{q}}(0) = \dot{\bar{q}}_0$$

where $\bar{q}_0, \dot{\bar{q}}_0$ are fixed vectors.

It may be shown that this problem has a unique solution if:

$$\det M \neq 0$$

$\bar{P} = \bar{P}(t)$ is a continuous function



Leonard Euler

1707-1783



Jean Le Rond d'Alembert

1717 - 1783



Joseph-Louis Lagrange

1736 - 1813



John William Strutt Lord Rayleigh

1842 - 1919



Johann Peter Gustav Lejeune Dirichlet

1805 - 1859

82. We may now introduce the condition that the motion takes place in the immediate neighbourhood of a configuration of thoroughly stable equilibrium; T and F are then homogeneous quadratic functions of the velocities with coefficients which are to be treated as constant, and V is a similar function of the co-ordinates themselves, provided that (as we suppose to be the case) the origin of each co-ordinate is taken to correspond with the configuration of equilibrium. Moreover all three functions are essentially positive. Since terms of the form $dT/d\psi$ are of the second order of small quantities, the equations of motion become linear, assuming the form

$$\frac{d}{dt} \left(\frac{dT}{d\dot{\psi}} \right) + \frac{dF}{d\dot{\psi}} + \frac{dV}{d\psi} = \Psi \dots\dots\dots (1),$$

where under Ψ are to be included all forces acting on the system not already provided for by the differential coefficients of F and V .

The three quadratic functions will be expressed as follows:—

$$\left. \begin{aligned} T &= \frac{1}{2} a_{11} \dot{\psi}_1^2 + \frac{1}{2} a_{22} \dot{\psi}_2^2 + \dots + a_{12} \dot{\psi}_1 \dot{\psi}_2 + \dots \\ F &= \frac{1}{2} b_{11} \psi_1^2 + \frac{1}{2} b_{22} \psi_2^2 + \dots + b_{12} \psi_1 \psi_2 + \dots \\ V &= \frac{1}{2} c_{11} \psi_1^2 + \frac{1}{2} c_{22} \psi_2^2 + \dots + c_{12} \psi_1 \psi_2 + \dots \end{aligned} \right\} \dots\dots\dots (2),$$

where the coefficients a, b, c are constants.

From "The theory of Sound. Vol 1."

LECTURE 5

UNDAMPED VIBRATIONS

Non-gyroscopic systems
Modal decomposition

UNDAMPED VIBRATIONSn-degree-of-freedom-systems12. Non-gyroscopic systems

We here assume that:

$$\underline{B} = \underline{0} \quad (12.1)$$

We then have the vibration equation:

$$\underline{M} \ddot{\underline{q}} + \underline{K} \underline{q} = \underline{\bar{p}} \quad (12.2)$$

$$\underline{M}^T = \underline{M} ; \text{ strictly positive definite}$$

$$\underline{K}^T = \underline{K} ; \text{ positive semi-definite}$$

We first look at free vibrations,
i.e. no external force:

$$\underline{\bar{p}} = \underline{0}$$

and then:

$$\underline{M} \ddot{\underline{q}} + \underline{K} \underline{q} = \underline{0} \quad (12.3)$$

We try the following "ansatz":

$$\underline{\bar{q}} = \underline{\bar{q}}(t) = \underline{\bar{x}} \phi(t) \quad (12.4)$$

where $\underline{\bar{x}}$ is a constant vector. All the coordinates $q_1, \dots, q_n$ will then have a synchronous time dependence given by the function:

$$\phi = \phi(t)$$

We seek $\underline{\bar{x}}$ and ϕ so that (12.3) is satisfied,

$$\underline{\bar{q}} = \underline{\bar{x}} \phi \Rightarrow \ddot{\underline{\bar{q}}} = \underline{\bar{x}} \ddot{\phi} = 0$$

$$\ddot{\phi} \underline{M} \underline{\bar{x}} + \phi \underline{K} \underline{\bar{x}} = \underline{0}$$

$$\underline{K} \underline{\bar{x}} = - \frac{\ddot{\phi}}{\phi} \underline{M} \underline{\bar{x}} \Rightarrow (\underline{\bar{x}} \neq \underline{0})$$

$$- \frac{\ddot{\phi}}{\phi} = \lambda = \text{constant} \in \mathbb{R}$$

$$\ddot{\phi} + \lambda \phi = 0 \quad (12.5)$$

and

$$\underline{K} \underline{\bar{x}} = \lambda \underline{M} \underline{\bar{x}} \quad (12.6)$$

pre-multiply with $\underline{\bar{x}}^T$!

$$\underline{\bar{x}}^T \underline{K} \underline{\bar{x}} = \lambda \underline{\bar{x}}^T \underline{M} \underline{\bar{x}} \Rightarrow (\underline{\bar{x}} \neq \underline{0})$$

$$\lambda = \frac{\underline{\bar{x}}^T \underline{K} \underline{\bar{x}}}{\underline{\bar{x}}^T \underline{M} \underline{\bar{x}}} \geq 0 \quad (12.7)$$

and then we may choose $\omega \in \mathbb{R}$ such that

$$\lambda = \omega^2 \quad (\omega = \pm \sqrt{\lambda}) \quad (12.8)$$

and (12.6) is equivalent to

$$(-\omega^2 \underline{M} + \underline{K}) \underline{\bar{x}} = \underline{0} \quad (12.9)$$

This is a linear equation system in the unknowns $x_1, \dots, x_n$

$$\begin{pmatrix} -\omega^2 m_{11} + k_{11} & \dots & -\omega^2 m_{1n} + k_{1n} \\ -\omega^2 m_{21} + k_{21} & \dots & -\omega^2 m_{2n} + k_{2n} \\ \vdots & & \vdots \\ -\omega^2 m_{n1} + k_{n1} & \dots & -\omega^2 m_{nn} + k_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

There is a solution $\bar{x} \neq \bar{0}$ if and only if

$$\| \det(-\omega^2 \underline{M} + \underline{K}) = 0 \quad (12.10)$$

This is the so-called characteristic equation. It may be written:

$$\| c_n (\omega^2)^n + c_{n-1} (\omega^2)^{n-1} + \dots + c_0 = 0 \quad (12.11)$$

where c_n are real constants:

$$\begin{aligned} c_n &= (-1)^n \det \underline{M} \neq 0 \\ c_0 &= \det \underline{K} \end{aligned} \quad (12.12)$$

We have thus shown that:

$$\| \bar{q} = \bar{x} \phi \quad (\bar{x} \neq \bar{0}) \text{ is a solution to (12.3)}$$

$\Downarrow$

$$\ddot{\phi} + \omega^2 \phi = 0 \quad (12.13)$$

$$(-\omega^2 \underline{M} + \underline{K}) \bar{x} = 0$$

ω satisfies the charact. equation.

The solution of (12.13) is given by; if

$$\underline{\omega^2 > 0}$$

$$\| \phi = \phi(t) = \alpha \cos \omega t + \beta \sin \omega t \quad (12.14)$$

where $\alpha, \beta \in \mathbb{R}$ are arbitrary constants

$$\underline{\omega^2 = 0}$$

$$\| \phi = \phi(t) = \alpha + \beta t \quad (12.15)$$

$\alpha, \beta \in \mathbb{R}$ arbitrary constants.

We have thus the following "free vibrational modes".

$$\| \bar{q} = \bar{x} (\alpha \cos \omega t + \beta \sin \omega t), \quad \omega^2 > 0$$

$$\| \bar{q} = \bar{x} (\alpha + \beta t), \quad \omega^2 = 0$$

Note that the second mode is actually not a vibrating mode. It represents a so-called "rigid-body mode".

We now formulate the main result
for free undamped, non-gyroscopic
vibrations:

To the (generalized) eigenvalue problem
(12.6) there exists n linearly inde-
pendent real eigenvectors:

$$\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$$

and corresponding real eigenvalues

$$0 \leq \omega_1^2 \leq \omega_2^2 \leq \dots \leq \omega_n^2$$

satisfying

$$(-\omega_i^2 \underline{M} + \underline{K}) \bar{x}_i = \bar{0}, \quad i=1, \dots, n$$

The eigenvectors have the following
orthogonality relations:

$$\begin{aligned} \bar{x}_i^T \underline{M} \bar{x}_j &= 0 \\ \bar{x}_i^T \underline{K} \bar{x}_j &= 0 \end{aligned} \quad i \neq j, \quad i=1, \dots, n \quad (12.16)$$

The eigenvalues satisfy:

$$\omega_i^2 = \frac{\bar{x}_i^T \underline{K} \bar{x}_i}{\bar{x}_i^T \underline{M} \bar{x}_i}, \quad i=1, \dots, n \quad (12.17)$$

and consequently

$$\omega_i^2 \geq 0$$

Note that the eigenvalues need not be
pairwise different. We may have mul-
tipl eigenvalues.

Note also that (12.16) implies that
the vectors $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ are linearly
independent.

Proof: We first assume that the
eigenvalues are single:

$$\omega_1^2 < \omega_2^2 < \dots < \omega_n^2 \quad (12.18)$$

If $\bar{x}_i$ is an eigenvector correspon-
ding to ω_i^2 then

$$(-\omega_i^2 \underline{M} + \underline{K}) \bar{x}_i = \bar{0}, \quad \bar{x}_i \neq \bar{0} \quad (12.19)$$

We do not, at this point, know if $\bar{x}_i$ is a real vector so we put

$$\bar{x}_i = \bar{v}_i + i\bar{w}_i \quad (12.20)$$

$\bar{v}_i, \bar{w}_i$ are real vectors and $i^2 = -1$ ("imaginary unit")

From (12.19) we get by premultiplication with $\bar{x}_i^* = \bar{v}_i^T - i\bar{w}_i^T$:

$$\begin{aligned} \bar{x}_i^* (-\omega_i^2 \underline{M} + \underline{K}) \bar{x}_i &= \\ (\bar{v}_i^T - i\bar{w}_i^T)(-\omega_i^2 \underline{M} + \underline{K})(\bar{v}_i + i\bar{w}_i) &= \\ -\omega_i^2 (\bar{v}_i^T \underline{M} \bar{v}_i + \bar{w}_i^T \underline{M} \bar{w}_i) + \bar{v}_i^T \underline{K} \bar{v}_i + \\ \bar{w}_i^T \underline{K} \bar{w}_i + i \{-\omega_i^2 (\bar{v}_i^T \underline{M} \bar{w}_i - \bar{w}_i^T \underline{M} \bar{v}_i) + \\ (\bar{v}_i^T \underline{K} \bar{w}_i - \bar{w}_i^T \underline{K} \bar{v}_i)\} &= \bar{0} \end{aligned} \quad (12.21)$$

But since $\underline{M}$ and $\underline{K}$ are symmetric

$$\begin{aligned} \bar{v}_i^T \underline{M} \bar{w}_i &= \bar{w}_i^T \underline{M} \bar{v}_i \\ \bar{v}_i^T \underline{K} \bar{w}_i &= \bar{w}_i^T \underline{K} \bar{v}_i \end{aligned} \quad (12.22)$$

and consequently the imaginary

in (12.21) is zero. Thus

$$\begin{aligned} -\omega_i^2 (\bar{v}_i^T \underline{M} \bar{v}_i + \bar{w}_i^T \underline{M} \bar{w}_i) + \bar{v}_i^T \underline{K} \bar{v}_i + \\ \bar{w}_i^T \underline{K} \bar{w}_i &= \bar{0} \quad \Rightarrow \end{aligned}$$

$$\omega_i^2 = \frac{\bar{v}_i^T \underline{K} \bar{v}_i + \bar{w}_i^T \underline{K} \bar{w}_i}{\bar{v}_i^T \underline{M} \bar{v}_i + \bar{w}_i^T \underline{M} \bar{w}_i} \quad (12.23)$$

which is a non-negative real number.

From (12.19) we get

$$-\omega_i^2 \underline{M} \bar{v}_i + \underline{K} \bar{v}_i = i(-\omega_i^2 \underline{M} \bar{w}_i + \underline{K} \bar{w}_i) = \bar{0} \quad (12.24)$$

since the left member is a real vector whereas the right is imaginary.

Since $\bar{x}_i \neq \bar{0}$ at least one of $\bar{v}_i, \bar{w}_i$

$\neq \bar{0}$ and according to (12.24) a

real eigenvector corresponding to

ω_i^2 . We may choose $\bar{x}_i$ real and

(consequently) $\bar{w}_i = \bar{0}$. From (12.23) we

then get

$$\omega_i^2 = \frac{\bar{x}_i^T \underline{K} \bar{x}_i}{\bar{x}_i^T \underline{M} \bar{x}_i}$$

Finally we show the orthogonality relations.

$$\begin{aligned} (-\omega_i^2 \underline{M} + \underline{K}) \bar{x}_i &= \bar{0} \\ (-\omega_j^2 \underline{M} + \underline{K}) \bar{x}_j &= \bar{0} \end{aligned} \Rightarrow \begin{pmatrix} \text{multiply} \\ \text{with } \bar{x}_j^T \\ \text{and } \bar{x}_i^T \\ \text{resp.} \end{pmatrix}$$

$$\omega_i^2 \bar{x}_j^T \underline{M} \bar{x}_i = \bar{x}_j^T \underline{K} \bar{x}_i \Rightarrow \text{(subtract)}$$

$$\omega_j^2 \bar{x}_i^T \underline{M} \bar{x}_j = \bar{x}_i^T \underline{K} \bar{x}_j$$

$$(\omega_i^2 - \omega_j^2) \bar{x}_i^T \underline{M} \bar{x}_j = \bar{0} \Rightarrow (i \neq j)$$

$$\bar{x}_i^T \underline{M} \bar{x}_j = 0 \quad \text{since } \omega_i^2 \neq \omega_j^2$$

and the proof is completed for the case of single eigenvalues. A proof in the case of multiple eigenvalues is given in Geradin & Rixen (p. 48)

□

Note that if $\bar{x}$ is an eigenvector then so is $\lambda \bar{x}$, $\forall \lambda \in \mathbb{R}$ which means that eigenvectors are determined only up to a constant factor. We call $\bar{x}$

(or $\lambda \bar{x}$) a mode shape vector.

We now introduce the following notation:

$$\begin{aligned} \bar{x}_i^T \underline{M} \bar{x}_i &= \mu_i & i=1, \dots, n \\ \bar{x}_i^T \underline{K} \bar{x}_i &= \kappa_i & i=1, \dots, n \end{aligned} \quad (12.25) \quad *$$

where μ_i is called modal mass^{**} and κ_i modal stiffness. These quantities are not uniquely determined by the mode shape, since

$$\bar{x}_i \rightarrow \lambda \bar{x}_i \Rightarrow \mu_i \rightarrow \lambda^2 \mu_i, \quad \kappa_i \rightarrow \lambda^2 \kappa_i$$

Their quotient is however unique since:

$$\frac{\kappa_i}{\mu_i} = \omega_i^2 \quad (12.26)$$

The modal matrix is defined by:

$$\bar{X} = [\bar{x}_1, \dots, \bar{x}_n] \quad (12.27)$$

* κ_i is used in Geradin & Rixen

** or "generalized mass".

Since $\bar{x}_1, \dots, \bar{x}_n$ are linearly independent
 $\underline{X}$ is nonsingular and

$$|\det \underline{X}| \neq 0 \quad (12.28)$$

By a change of coordinates: $\bar{q} \rightarrow \bar{\eta}$
 defined by:

$$\bar{q} = \underline{X}\bar{\eta} = \bar{x}_1\eta_1 + \dots + \bar{x}_n\eta_n \quad (12.29)$$

where the normal coordinates*

$$\bar{\eta} = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix}$$

is a new set of coordinates. We
 have the following transformations of
 mass and stiffness matrices:

$$\underline{M} \rightarrow \underline{\mu} = \underline{X}^T \underline{M} \underline{X} = [\mu_{ij}] \quad (12.30)$$

$$\underline{K} \rightarrow \underline{x} = \underline{X}^T \underline{K} \underline{X} = [x_{ij}]$$

(cf. formulas (11.20))

* or principal coordinates.

From the orthogonality conditions (12.16)
 we get:

$$\mu_{ij} = \begin{cases} \mu_i & i=j \\ 0 & i \neq j \end{cases} \quad (12.31)$$

$$x_{ij} = \begin{cases} x_i & i=j \\ 0 & i \neq j \end{cases}$$

or expressed in another way:

$$\underline{\mu} = \text{diag}(\mu_1, \dots, \mu_n), \quad \mu_i > 0$$

$$\underline{x} = \text{diag}(x_1, \dots, x_n), \quad x_i \geq 0$$

where diag = diagonal matrix.

The vibrational equation expressed
 in the new coordinates now reads:

$$\underline{\mu} \ddot{\bar{\eta}} + \underline{x} \bar{\eta} = \underline{X}^T \bar{P} \quad (12.32)$$

and for the free vibrations ($\bar{P} = \bar{0}$)

$$\underline{\mu} \ddot{\bar{\eta}} + \underline{x} \bar{\eta} = \bar{0} \quad (12.33)$$

or in component form

$$M_i \ddot{\eta}_i + c_i \dot{\eta}_i = 0$$

= 0 (by division with $M_i > 0$)

$$\ddot{\eta}_i + \omega_i^2 \eta_i = 0 \quad (12.34)$$

With the solution (for degree of freedom $n \geq 1$)

$$\omega_i^2 = 0 :$$

$$\eta_i = \alpha_i + \beta_i t$$

$$(12.35)$$

$$\omega_i^2 > 0 :$$

$$\eta_i = \alpha_i \cos \omega_i t + \beta_i \sin \omega_i t$$

$$(12.36)$$

We thus have two types of motion:

rigid-body modes ($\omega_i^2 = 0$):

$$\underline{q} = \underline{x}_i (\alpha_i + \beta_i t)$$

$$(12.37)$$

and vibrational modes ($\omega_i^2 > 0$)

$$\underline{q} = \underline{x}_i (\alpha_i \cos \omega_i t + \beta_i \sin \omega_i t) \quad (12.38)$$

The general solution to the free vibration problem (12.3) may now be written:

be written:

$$\underline{q} = \sum_{i=1}^n \underline{x}_i (\alpha_i + \beta_i t) + \sum_{i=1}^n \underline{x}_i (\alpha_i \cos \omega_i t + \beta_i \sin \omega_i t)$$

$$\omega_i^2 = 0$$

$$(12.39)$$

$$\sum_{i=1}^n \underline{x}_i (\alpha_i \cos \omega_i t + \beta_i \sin \omega_i t)$$

$$\omega_i^2 > 0$$

where α_i, β_i are 2n real arbitrary constants. These constants are

determined by the initial conditions:

$$\underline{q}(0) = \underline{q}_0$$

$$\dot{\underline{q}}(0) = \dot{\underline{q}}_0$$

$$(12.40)$$

Let us assume that the first m , ($m \leq \min(6, n)$), modes are rigid-body modes and thus

$$\omega_i^2 = 0 \quad i = 1, \dots, m$$

and that the following $n-m$ modes are vibrational modes:

$$\omega_i^2 > 0 \quad i = m+1, \dots, n$$

We then have:

$$\bar{q} = \sum_{i=1}^m \bar{x}_i (\alpha_i + \beta_i t) + \sum_{i=m+1}^n \bar{x}_i (\alpha_i \cos \omega_i t + \beta_i \sin \omega_i t) \quad (12.41)$$

$\Rightarrow$

$$\bar{q}(0) = \sum_{i=1}^m \bar{x}_i \alpha_i + \sum_{i=m+1}^n \bar{x}_i \alpha_i = \bar{q}_0 \quad (12.42)$$

$$\dot{\bar{q}}(0) = \sum_{i=1}^m \bar{x}_i \beta_i + \sum_{i=m+1}^n \bar{x}_i \omega_i \beta_i = \dot{\bar{q}}_0$$

and from this we may conclude that:

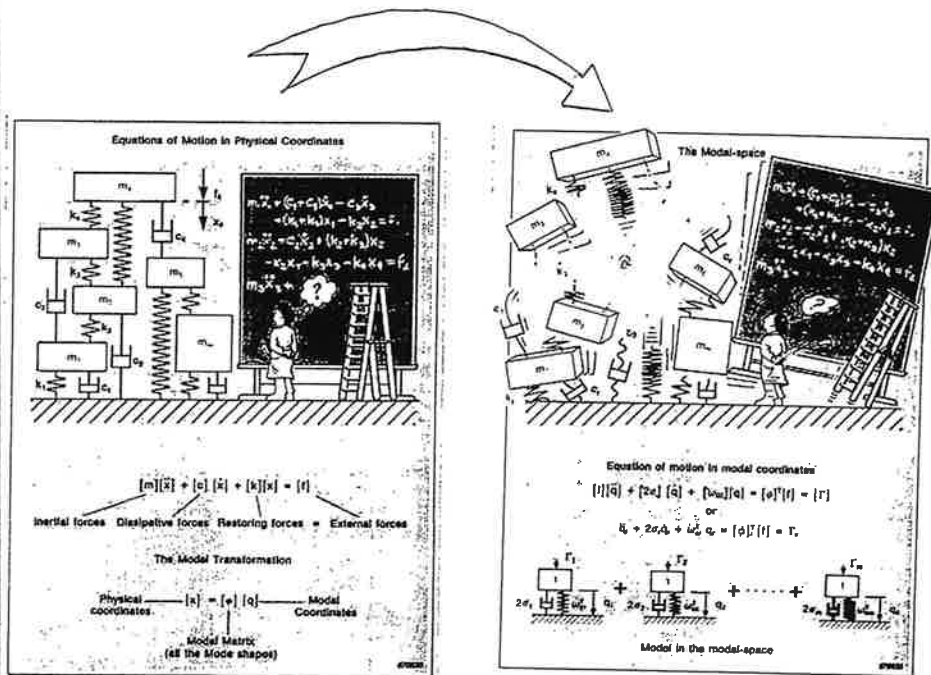
$$\alpha_i = \frac{\bar{x}_i^T M \bar{q}_0}{\mu_i} \quad i = 1, \dots, m$$

$$\beta_i = \begin{cases} \frac{\bar{x}_i^T M \dot{\bar{q}}_0}{\mu_i} & i = 1, \dots, m \\ \frac{\bar{x}_i^T M \dot{\bar{q}}_0}{\mu_i \omega_i} & i = m+1, \dots, n \end{cases} \quad (12.43)$$

and the solution to the initial value problem:

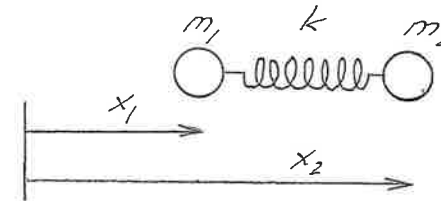
$$\bar{q} = \bar{q}(t) = \left(\sum_{i=1}^m \frac{\bar{x}_i \bar{x}_i^T M}{\mu_i} + \sum_{i=m+1}^n \frac{\bar{x}_i \bar{x}_i^T M}{\mu_i} \cos \omega_i t \right) \bar{q}_0 + \left(\sum_{i=1}^m \frac{\bar{x}_i \bar{x}_i^T M}{\mu_i} t + \sum_{i=m+1}^n \frac{\bar{x}_i \bar{x}_i^T M}{\mu_i \omega_i} \sin \omega_i t \right) \dot{\bar{q}}_0 \quad (12.44)$$

THE MODAL DECOMPOSITION



(From Brüel & Kjaer, Structural testing,
www.bksv.com)

Ex 13 Two-particle system:



$$T = \frac{1}{2} (m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2)$$

$$V = \frac{1}{2} k (x_2 - x_1)^2$$

$$L = T - V \Rightarrow \frac{\partial L}{\partial \dot{x}_1} = m_1 \dot{x}_1, \quad \frac{\partial L}{\partial \dot{x}_2} = m_2 \dot{x}_2$$

$$\frac{\partial L}{\partial x_1} = -k(x_2 - x_1), \quad \frac{\partial L}{\partial x_2} = k(x_2 - x_1)$$

L's eq.:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = m_1 \ddot{x}_1 + kx_1 - kx_2 = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = m_2 \ddot{x}_2 - kx_2 + kx_1 = 0$$

or in matrix form:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{M} \ddot{\underline{x}} + \underline{K} \underline{x} = \underline{0}$$

Eigenvalue problem:

$$(-\omega^2 \underline{M} + \underline{K}) \underline{u} = \underline{0}, \quad \underline{u} \neq \underline{0}$$

char. eq.

$$\det(-\omega^2 \underline{M} + \underline{K}) = \det \begin{pmatrix} -\omega^2 m_1 + k & -k \\ -k & -\omega^2 m_2 + k \end{pmatrix} =$$

$$(-\omega^2 m_1 + k)(-\omega^2 m_2 + k) - k^2 = \omega^4 m_1 m_2 -$$

$$\omega^2 k(m_1 + m_2) = 0 \quad \Rightarrow$$

$$\omega_1^2 = 0 \quad (\text{rigid-body mode})$$

$$\omega_2^2 = \frac{k}{\frac{m_1 m_2}{m_1 + m_2}} = \frac{k}{m_r}, \quad m_r = \frac{m_1 m_2}{m_1 + m_2}$$

Mode forms:

$$\omega_1^2 = 0 : \underline{K} \underline{u} = \underline{0} \Rightarrow k u_1 - k u_2 = 0$$

$$\Rightarrow u_1 = u_2 \Rightarrow \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (\text{rigid-body mode})$$

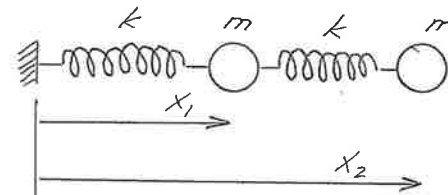
$$\omega_2^2 = \frac{k}{m_r} : (-\omega_2^2 \underline{M} + \underline{K}) \underline{u} = \underline{0} \Rightarrow$$

$$\left(-\frac{m_1 + m_2}{m_2} k + k\right) u_1 - k u_2 = 0 \Rightarrow$$

$$-\frac{m_1}{m_2} k u_1 - k u_2 = 0 \Rightarrow u_2 = -\frac{m_1}{m_2} u_1$$

$$\Rightarrow \underline{u} = \begin{pmatrix} 1 \\ -\frac{m_1}{m_2} \end{pmatrix}$$

#

Ex 14 Two-particle system.

$$T = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2), \quad V = \frac{1}{2} k (x_2 - x_1)^2 +$$

$$\frac{1}{2} k x_1^2, \quad L = T - V$$

$$\frac{\partial L}{\partial \dot{x}_1} = m \dot{x}_1, \quad \frac{\partial L}{\partial x_1} = -k(x_1 - x_2) - k x_1$$

$$\frac{\partial L}{\partial \dot{x}_2} = m \dot{x}_2, \quad \frac{\partial L}{\partial x_2} = -k(x_2 - x_1)$$

L's eq.:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) - \frac{\partial L}{\partial x_1} = m \ddot{x}_1 + 2k x_1 - k x_2 = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) - \frac{\partial L}{\partial x_2} = m \ddot{x}_2 - k x_1 + k x_2 = 0$$

or in matrix form:

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{M} \ddot{\underline{x}} + \underline{K} \underline{x} = \underline{0}$$

Eigenvalueproblem

$$(-\omega^2 \underline{M} + \underline{K}) \underline{u} = \underline{0}, \quad \underline{u} \neq \underline{0}$$

char. eq.

$$\det(-\omega^2 \underline{M} + \underline{K}) = \det \begin{pmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + k \end{pmatrix} =$$

$$(-\omega^2 m + 2k)(-\omega^2 m + k) - k^2 = 0 \Rightarrow$$

$$\omega^4 m^2 - \omega^2 3km + k^2 = 0 \Rightarrow$$

$$\omega_{1,2}^2 = \frac{3}{2} \frac{k}{m} \pm \sqrt{\frac{9}{4} \frac{k^2}{m^2} - \frac{k^2}{m^2}} = \left(\frac{3}{2} \pm \frac{\sqrt{5}}{2} \right) \frac{k}{m}$$

Mode forms:

$$\omega_1^2 = \left(\frac{3}{2} - \frac{\sqrt{5}}{2} \right) \frac{k}{m} : (-\omega_1^2 \underline{M} + \underline{K}) \underline{u} = \underline{0} \Rightarrow$$

$$\left(-\left(\frac{3}{2} - \frac{\sqrt{5}}{2} \right) k + 2k \right) u_1 - k u_2 = 0 \Rightarrow$$

$$u_2 = \frac{\sqrt{5}+1}{2} u_1 \Rightarrow$$

$$\underline{u} = \begin{pmatrix} 1 \\ \frac{\sqrt{5}+1}{2} \end{pmatrix}$$

$$\omega_2^2 = \left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right) k : (-\omega_2^2 \underline{M} + \underline{K}) \underline{u} = \underline{0} \Rightarrow$$

$$\left(-\left(\frac{3}{2} + \frac{\sqrt{5}}{2} \right) k + 2k \right) u_1 - k u_2 = 0 \Rightarrow$$

$$u_2 = -\frac{\sqrt{5}-1}{2} u_1 \Rightarrow$$

$$\underline{u} = \begin{pmatrix} 1 \\ -\frac{\sqrt{5}-1}{2} \end{pmatrix}$$

#

LECTURE 6

UN-DAMPED VIBRATIONS

Forced harmonic response
 Mechanical admittance
 Resonance and anti-resonance

A note on linear algebra

Let $\bar{e}_1, \dots, \bar{e}_n$ be an arbitrary basis in $\mathbb{R}^n$ (not necessarily orthonormal). The reciprocal basis $\bar{e}'_1, \dots, \bar{e}'_n$ is defined by the conditions

$$\bar{e}'_i \cdot \bar{e}_j = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad (12.45)$$

Now if $\bar{u}$ is a vector in $\mathbb{R}^n$ then

$$\bar{u} = \sum_{i=1}^n \bar{e}_i u^i, \quad u^i = (\bar{e}'^i)^T \bar{u} \quad (12.46)$$

i.e. the components u^i are found by projection on the reciprocal basis.

Now let $\bar{e}_i = \bar{x}_i$ (mode shapes) then we find that $\bar{x}'^i = \frac{M \bar{x}_i}{\mu_i}$, and

$$\bar{u} = \sum_{i=1}^n \bar{x}_i \left(\frac{M \bar{x}_i}{\mu_i} \right)^T \bar{u} = \left(\sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T M}{\mu_i} \right) \bar{u}$$

which means that

$$\sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i} = \underline{1} = \text{diag}(1 \dots 1) \quad (12.47)$$

is the unit matrix. We also have

$$\sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T \underline{K}}{\kappa_i} = \underline{1} \quad (12.48)$$

show that:

$$\underline{M}^{-1} = \sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T}{\mu_i} \quad (12.49)$$

$$\underline{K}^{-1} = \sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T}{\kappa_i} \quad \text{if } \kappa_i > 0$$

Note that the unit matrix $\underline{1}$ may be written

$$\underline{1} = \sum_{i=1}^n \bar{x}_i (\bar{x}_i^i)^T = \sum_{i=1}^n \bar{x}_i^i (\bar{x}_i^i)^T$$

Forced harmonic response

We now assume that the external force is harmonic, i.e.

$$\bar{P} = \bar{P}(t) = \bar{P}_0 \cos \omega t \quad (12.50)$$

where $\bar{P}_0$ is a constant vector. The equation of motion may then be written:

$$\underline{M} \ddot{\bar{q}} + \underline{K} \bar{q} = \bar{P} = \bar{P}_0 \cos \omega t \quad (12.51)$$

We try the "ansatz":

$$\bar{q} = \bar{q}(t) = \bar{x} \cos \omega t = v$$

$$\ddot{\bar{q}} = -\omega^2 \bar{x} \cos \omega t = -\omega^2 \bar{q} = v$$

$$(-\omega^2 \underline{M} + \underline{K}) \bar{x} \cos \omega t = \bar{P}_0 \cos \omega t, \quad \forall t = v$$

$$(-\omega^2 \underline{M} + \underline{K}) \bar{x} = \bar{P}_0 \quad (12.52)$$

using the dynamical stiffness (see (11.24))

$$\underline{Z}(s) = \underline{M} s^2 + \underline{K}$$

We have

$$-\omega^2 \underline{M} + \underline{K} = \underline{Z}(i\omega)$$

and (12.52) may be written:

$$\underline{Z}(i\omega) \bar{X} = \bar{P}_0, \quad \bar{P}_0 = \text{external force}$$

$$\det \underline{Z}(i\omega) \neq 0 \Rightarrow \bar{X} = \underline{Z}(i\omega)^{-1} \bar{P}_0 \quad (12.53)$$

$$\det \underline{Z}(i\omega) = 0 \Rightarrow \bar{X} \text{ may not exist}$$

We introduce the mechanical admittance*

$$\underline{A}(s) = \underline{Z}(s)^{-1} \quad \text{if } \det \underline{Z}(s) \neq 0 \quad (12.54)$$

and we may write:

$$\bar{X} = \underline{A}(i\omega) \bar{P}_0 = (-\omega^2 \underline{M} + \underline{K})^{-1} \bar{P}_0, \quad \omega \neq \omega_k \quad (12.55)$$

$$\text{With } \underline{A}(i\omega) = [a_{ij}(\omega)] = \underline{A}(i\omega)^T$$

$$x_i = \sum_{j=1}^n a_{ij}(\omega) p_{0j}$$

$$\text{With } \bar{P}_0 = (0, 0, \dots, \overset{k^{\text{th position}}}{1}, \dots, 0)^T \Rightarrow$$

$$x_i = a_{ik}$$

* or transfer function: $\bar{P}_0 \rightarrow \bar{X}$

Spectral expansion:

$$\bar{X} = \sum_{i=1}^n \alpha_i \bar{x}_i \quad (12.56)$$

$\bar{x}_1, \dots, \bar{x}_n$ eigenmodes. Then

$$\bar{P}_0 = (-\omega^2 \underline{M} + \underline{K}) \bar{X} = \sum_{i=1}^n \alpha_i (-\omega^2 \underline{M} + \underline{K}) \bar{x}_i$$

$$\bar{x}_j^T \bar{P}_0 = \sum_{i=1}^n \alpha_i (-\omega^2 \bar{x}_j^T \underline{M} \bar{x}_i + \bar{x}_j^T \underline{K} \bar{x}_i) =$$

$$\alpha_j (-\omega^2 \mu_j + \gamma_j) \Rightarrow$$

$$\alpha_j = \frac{\bar{x}_j^T \bar{P}_0}{\gamma_j - \omega^2 \mu_j} = \frac{\bar{x}_j^T \bar{P}_0}{(\omega_j^2 - \omega^2) \mu_j}, \quad \omega \neq \omega_j$$

($\omega = \omega_j \Rightarrow \bar{x}_j^T \bar{P}_0 = 0$) and then:

$$\bar{X} = \left(\sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T}{(\omega_i^2 - \omega^2) \mu_i} \right) \bar{P}_0 \quad (12.57)$$

Consequently:

$$\underline{A}(i\omega) = \sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T}{(\omega_i^2 - \omega^2) \mu_i} \quad \omega \neq \omega_i \quad (12.58)$$

or in component form:

$$a_{ij}(\omega) = \sum_{k=1}^n \frac{(\bar{x}_k)_i (\bar{x}_k)_j}{(\omega_k^2 - \omega^2) \mu_k} \quad (12.59)$$

where $(\bar{x}_k)_i = x_{ik}$ is the i th component of the mode shape vector $\bar{x}_k$.

Assume that $\det \underline{K} \neq 0$ (i.e. $\kappa_k > 0, \forall k$) "no rigid-body modes". Then

$$\underline{A}(0) = \sum_{i=1}^n \frac{\bar{x}_i \bar{x}_i^T}{\kappa_i} = \underline{K}^{-1} \quad (12.60)$$

where we have used (12.49) and the fact that $\kappa_k = \omega_k^2 \mu_k, k = 1, \dots, n$.

Note that for the diagonal element a_{ii}

$$\frac{da_{ii}}{d\omega} = \sum_{k=1}^n \frac{2\omega (\bar{x}_k)_i^2}{(\omega_k^2 - \omega^2)^2 \mu_k} > 0 \quad (12.61)$$

$a_{ii} = a_{ii}(\omega)$ is thus an increasing function. Furthermore:

$$\lim_{\omega \rightarrow \omega_k^-} a_{ii} = +\infty \quad \lim_{\omega \rightarrow \omega_k^+} a_{ii} = -\infty, \quad \forall k$$

$$a_{ii}(0) = \sum_{k=1}^n \frac{(\bar{x}_k)_i^2}{\omega_k^2 \mu_k} = (\underline{K}^{-1})_{ii} > 0 \quad (12.62)$$

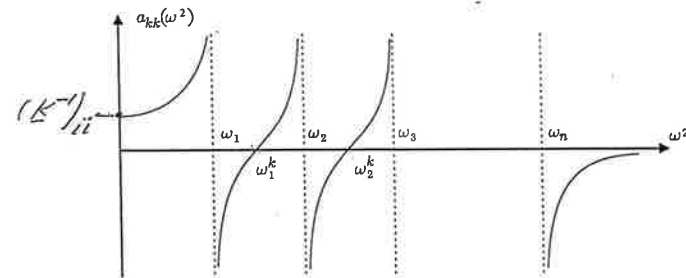


Figure 2.6.1
Principal dynamic influence coefficient
for a system with no rigid-body mode

The phenomena $a_{ii} \rightarrow \pm\infty$ as $\omega \rightarrow \omega_k$ is called resonance. In this case the external force

$$\bar{p} = \bar{s} \cos \omega t$$

has an angular frequency close to the eigenfrequency. We may call

$$0 < \omega_1 \leq \omega_2 \leq \dots \leq \omega_n$$

resonance frequencies. There are separated by antiresonance frequencies $\omega_i^a, i = 1, \dots, n-1$:

$$\omega_1 \leq \omega_1^a \leq \omega_2 \leq \omega_2^a \leq \dots \leq \omega_{n-1}^a \leq \omega_n$$

satisfying

$$a_{ij}(\omega_k^2) = 0 \quad k=1, \dots, n-1 \quad (12.63)$$

Note that the resonance frequencies are specific for the mechanical system whereas the antiresonances are depending on the particular degree of freedom q_i considered. Note that

$$a_{ij}(\omega) = (\underline{K}^{-1})_{ij} \frac{\prod_{k=1}^{n-1} (1 - (\frac{\omega}{\omega_k^2})^2)}{\prod_{k=1}^n (1 - (\frac{\omega}{\omega_k^2})^2)} \quad (12.64)$$

Now what if $\det \underline{K} = 0$? The only difference with the previous case ($\det \underline{K} \neq 0$) is that when $\omega^2 \rightarrow 0$:

$$\lim_{\omega^2 \rightarrow 0} a_{ij}(\omega) = -\infty \quad (12.65)$$

The number of antiresonance frequencies is equal to the number of resonance frequencies with $\omega_i^2 > 0$; i.e. equal to $n-m$.

Note that

$$\lim_{\omega^2 \rightarrow 0} \omega^2 a_{ij}(\omega) = - \sum_{k=1}^m \frac{(\bar{x}_k)_i^2}{\mu_k} < 0 \quad (12.66)$$

where $\bar{x}_1, \dots, \bar{x}_m$ are the rigid-body modes, ($0 \leq m \leq 6$).

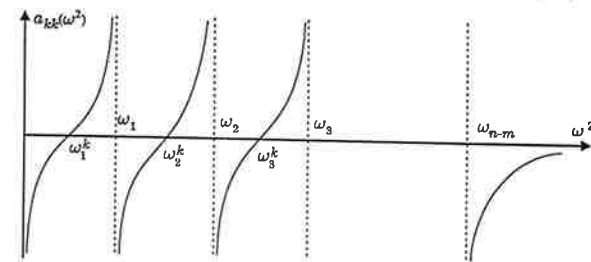


Figure 2.6.2
Principal dynamic influence coefficient
for a system with rigid-body modes

We also have

$$a_{ij}(\omega) = \frac{\lim_{\omega^2 \rightarrow 0} \omega^2 a_{ij}(\omega)}{\omega^2} \cdot \frac{\prod_{k=1}^{n-m} (1 - (\frac{\omega}{\omega_k^2})^2)}{\prod_{k=1}^n (1 - (\frac{\omega}{\omega_k^2})^2)} \quad (12.67)$$

where the product in the denominator contains resonance frequencies with $\omega_k^2 > 0$.

Another note on linear algebra.

Let $\underline{A}$ be an $n \times n$ -matrix with elements a_{ij} ,

$$\underline{A} = \begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & & & & & \\ a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & & & & & \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & & & & & \\ a_{n1} & \dots & a_{ni} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix} \quad i < j \quad (12.68)$$

We now introduce the sub-matrix $\underline{A}_{ij}$ defined by deleting the i :th row and j :th column.

$$\underline{A}_{ij} = \begin{pmatrix} a_{11} & \dots & a_{1i} & \dots & a_{1j} & \dots & a_{1n} \\ \vdots & & & & & & \\ a_{i1} & \dots & a_{ii} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & & & & & & \\ a_{j1} & \dots & a_{ji} & \dots & a_{jj} & \dots & a_{jn} \\ \vdots & & & & & & \\ a_{n1} & \dots & a_{ni} & \dots & a_{nj} & \dots & a_{nn} \end{pmatrix} \quad (12.69)$$

$\underline{A}_{ij}$ is an $(n-1) \times (n-1)$ - matrix.

The adjoint matrix to $\underline{A}$, $\text{adj} \underline{A}$ is now defined by

$$\| (\text{adj} \underline{A})_{ij} = (-1)^{i+j} \det \underline{A}_{ji} \quad (12.71)$$

with the property:

$$\| \underline{A} \text{adj}(\underline{A}) = \text{adj}(\underline{A}) \underline{A} = (\det \underline{A}) \underline{1} \quad (12.72)$$

Precisely if $\det \underline{A} \neq 0$ is $\underline{A}$ invertible and the inverse matrix is given by:

$$\| \underline{A}^{-1} = \frac{1}{\det \underline{A}} \text{adj} \underline{A} \quad (12.73)$$

$$\underline{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \text{adj} \underline{A} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}; \quad \underline{A} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$\Rightarrow \text{adj} \underline{A} = \begin{pmatrix} b_2 c_3 - b_3 c_2 & c_2 a_3 - c_3 a_2 & a_2 b_3 - a_3 b_2 \\ b_3 c_1 - b_1 c_3 & c_3 a_1 - c_1 a_3 & a_3 b_1 - a_1 b_3 \\ b_1 c_2 - b_2 c_1 & c_1 a_2 - c_2 a_1 & a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Consider the mechanical impedance:

$$\| \underline{Z} = \underline{Z}(s), \quad s \in \mathbb{C} \quad (12.74)$$

and take the adjoint matrix:

$$\| \underline{Q} = \text{adj} \underline{Z} = [q_{ij}] \quad (12.75)$$

$$q_{ij}(s) = (-1)^{i+j} \det \underline{Z}_{ji}(s)$$

where, for instance, $\underline{Z}_{ii} = \underline{Z}_{ii}(s)$ is the mechanical impedance of the modified system we get by fixating the degree of freedom q_i : $q_i \equiv 0$.

The eigenfrequencies of this system are calculated from the char. eq.

$$\| \det \underline{Z}_{ii}(i\omega) = 0 \Leftrightarrow q_{ii}(i\omega) = 0 \quad (12.76)$$

The admittance:

$$\| \underline{A}(s) = \underline{Z}^{-1}(s) = \frac{1}{p(s)} \underline{Q}(s), \quad p(s) \neq 0 \quad (12.77)$$

where

$$p = p(s) = \det \underline{Z}(s)$$

is called the characteristic polynomial

$$p(s) = \det \underline{M} s^{2n} + \dots + \det \underline{K} \quad (12.79)$$

i.e. a polynomial of degree $2n$. Note that

$$p(i\omega) = (-1)^n \det \underline{M} (\omega^2)^n + \dots + \det \underline{K} \quad (12.80)$$

We find that

$$\underline{A}(i\omega) = \frac{1}{p(i\omega)} \underline{Q}(i\omega), \quad p(i\omega) \neq 0 \quad (12.81)$$

$$p(i\omega) \neq 0 \Leftrightarrow \omega \neq \omega_1, \dots, \omega_n.$$

Then

$$a_{ii}(\omega) = \frac{1}{p(i\omega)} q_{ii}(i\omega) = \frac{q_{ii}(i\omega)}{p(i\omega)}$$

where

$$q(s) = \det \underline{Z}_{ii}(s)$$

is the characteristic polynomial

of the modified system ($x_i \equiv 0$). This polynomial is of degree $2n-2$ (in s).

The eigenfrequencies to the modified system $\tilde{\omega}$ are the solutions to:

$$q(i\tilde{\omega}) = 0 \quad (12.82)$$

This will result in $n-1$ roots $\tilde{\omega}_1, \dots, \tilde{\omega}_{n-1}$.

The antiresonances to the original system are given by (see (12.66)):

$$a_{ii}(\omega^a) = 0 \Leftrightarrow q(i\omega^a) = 0$$

and we may conclude:

$$\tilde{\omega}_i = \omega_i^a \quad i = 1, \dots, n-1 \quad (12.83)$$

This means that the anti-resonances of a mechanical system, corresponding to the degree of freedom q_i , are equal to the resonances of the modified system where $q_i \equiv 0$.

Note that from the general relation:

$$\underline{Z}(s) \underline{Q}(s) = \underline{Q}(s) \underline{Z}(s) = p(s) \underline{1}$$

we find, by choosing $s = i\omega_j$, $j=1, \dots, n$

$$\underline{Z}(i\omega_j) \underline{Q}(i\omega_j) = p(i\omega_j) \underline{1} = \underline{0}$$

and according to this every column-vector in $\underline{Q}(i\omega_j) = [\bar{q}_1 \dots \bar{q}_n]$, not equal to $\underline{0}$, is a mode shape corresponding to the eigenfrequency ω_j . We have:

$$\underline{Z}(i\omega_j) \bar{q}_k = \underline{0} \quad \forall k$$

It may be shown that if ω_j is a simple eigenvalue then

$$\underline{Q}(i\omega_j) \neq \underline{0}$$

which means that at least one of $\bar{q}_1, \dots, \bar{q}_n$ is not the zero vector!

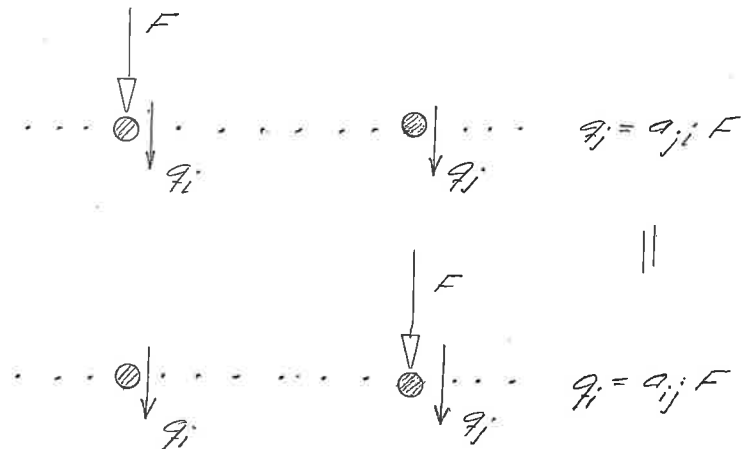
From (12.57) we have

$$\bar{x} = \underline{A}(i\omega) \bar{s}$$

where the admittance matrix is symmetric:

$$\underline{A}(i\omega)^T = \underline{A}(i\omega) \Leftrightarrow a_{ij}(\omega) = a_{ji}(\omega) \quad (12.83)$$

This is called reciprocity.



Note that (from (12.58)):

$$x_i = \bar{e}_i^T \underline{A}(i\omega) \bar{s} = \sum_{j=1}^n \frac{(\bar{e}_i^T \bar{x}_j)(\bar{x}_j^T \bar{s})}{(\omega_j^2 - \omega^2) \mu_j} \quad (12.84)$$

where $\bar{e}_i^T = (0, \dots, 1, \dots, 0)$

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$$(\bar{e}_i^T \bar{x}_j)(\bar{x}_j^T \bar{S}) = 0 \quad \forall j \Rightarrow x_j = 0$$

which means that if either $\bar{e}_i$
or $\bar{S}$ is orthogonal to every mode
shape then x_j is identically zero,
(for all ω), a kind of global
antiresonance.

LECTURE 7

UN-DAMPED VIBRATIONS

Eigenfrequency characterization

Rayleigh quotient

Minimax principle

13. Eigenfrequency characterization

The Rayleigh quotient for a mechanical system with mass-matrix $\underline{M}$ and stiffness-matrix $\underline{K}$ is defined by:

$$\mathcal{R}(\bar{x}) = \frac{\bar{x}^T \underline{K} \bar{x}}{\bar{x}^T \underline{M} \bar{x}}, \quad \bar{x} \neq \bar{0} \quad (13.1)$$

Obviously $\mathcal{R}(\bar{x}) \geq 0$ and $\mathcal{R}(\lambda \bar{x}) = \mathcal{R}(\bar{x})$,
(homogeneous of degree 0)

If we use normal coordinates:

$$\bar{x} = \underline{X} \bar{\eta} \quad (\underline{X} \text{ modal matrix})$$

then

$$\mathcal{R}(\bar{x}) = \frac{\sum_{i=1}^n x_i \eta_i^2}{\sum_{i=1}^n \mu_i \eta_i^2} = \frac{\sum_{i=1}^n \omega_i^2 \mu_i \eta_i^2}{\sum_{i=1}^n \mu_i \eta_i^2} \quad (13.2)$$

It follows that:

$$\max_{\bar{x} \neq \bar{0}} \mathcal{R}(\bar{x}) = \omega_n^2, \quad \min_{\bar{x} \neq \bar{0}} \mathcal{R}(\bar{x}) = \omega_1^2 \quad (13.3)$$

where

$$\omega_1^2 \leq \omega_2^2 \leq \dots \leq \omega_n^2$$

are the system eigenfrequencies. Thus

$$\omega_1^2 \leq R(\bar{x}) \leq \omega_n^2 \quad \forall \bar{x} \neq \bar{0} \quad (13.4)$$

Note that $R(\bar{x}_i) = \omega_i^2 \quad i = 1, \dots, n$.

We may perform another change of coordinates (to "normalized coordinates")

$$\bar{x} = \sqrt{\mu} \bar{\eta}$$

where $\sqrt{\mu} = \text{diag}(\sqrt{\mu_1}, \dots, \sqrt{\mu_n})$. This gives

$$R(\bar{x}) = \frac{\sum_{i=1}^n \omega_i^2 \bar{x}_i^2}{\sum_{i=1}^n \bar{x}_i^2} = \frac{\sum_{i=1}^n \omega_i^2 \bar{\eta}_i^2}{|\bar{\eta}|^2} \quad (13.5)$$

$\sum_{i=1}^n \omega_i^2 \bar{\eta}_i^2 = 1$ represents an ellipsoid in $\bar{\eta}$ -space with semi-axes: $a_i = \frac{1}{\omega_i}$

Since $R(\lambda \bar{x}) = R(\bar{x})$ we may restrict the study of the Rayleigh quotient to the ellipsoid in $\bar{\eta}$ -space:

$$E = \left\{ \bar{\eta} : \sum_{i=1}^n \omega_i^2 \bar{\eta}_i^2 = 1 \right\}$$

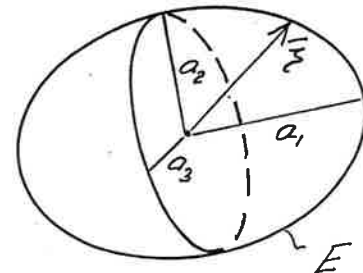
Then:

$$\omega_1^2 = \min_{\bar{x} \neq \bar{0}} R(\bar{x}) = \min_{\bar{\eta} \in E} \frac{1}{|\bar{\eta}|^2} =$$

$$\frac{1}{\max_{\bar{\eta} \in E} |\bar{\eta}|^2} = \frac{1}{a_1^2} \quad (13.6)$$

$$\omega_n^2 = \max_{\bar{x} \neq \bar{0}} R(\bar{x}) = \max_{\bar{\eta} \in E} \frac{1}{|\bar{\eta}|^2} =$$

$$\frac{1}{\min_{\bar{\eta} \in E} |\bar{\eta}|^2} = \frac{1}{a_n^2}$$



Consider two mechanical systems:

$S: \underline{M}, \underline{K}$ eigfreq. $\omega_1^2 \leq \omega_2^2 \leq \dots \leq \omega_n^2$

$S': \underline{M}, \underline{K}'$ eigfreq. $\omega_1'^2 \leq \omega_2'^2 \leq \dots \leq \omega_n'^2$

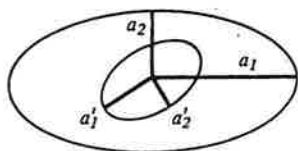
with the same mass-matrix. We say that S' is more rigid (stiffer) than S if:

$$\| \bar{x}^T \underline{K}' \bar{x} \geq \bar{x}^T \underline{K} \bar{x}, \quad \forall \bar{x} \quad (13.7)$$

This means that the potential energy for S' is not lesser than the potential energy for S . Then

$$\| \omega_i'^2 \geq \omega_i^2 \quad (13.8)$$

i.e. higher stiffness gives higher eigenfrequencies. This result follows easily from a comparison of ellipsoids.



We have now characterized ω_1^2 and ω_n^2 . The eigenfrequencies in between may also be given as solutions to a optimization problem, now with constraint conditions. We introduce the linear subspaces:

$$\mathcal{I}_k = \{ \bar{x}_1 \bar{x}_2 \dots \bar{x}_k \}$$

$$\mathcal{I}_k^\perp = \{ \bar{x}_{k+1} \dots \bar{x}_n \}$$

where $\{ \bar{x}_1 \bar{x}_2 \dots \bar{x}_k \}$ denotes the linear space spanned by $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k$. Note that:

$$\mathcal{I}_k^\perp = \{ \bar{x} : \bar{x}^T \underline{M} \bar{x}_j = 0, \quad \forall j = 1, \dots, k \}$$

Then

$$\| \omega_k^2 = \min_{\bar{x} \in \mathcal{I}_{k-1}^\perp} \mathcal{R}(\bar{x}) = \max_{\bar{x} \in \mathcal{I}_k} \mathcal{R}(\bar{x}) \quad (13.9)$$

The problem with this characterization is that we need to know the mode

shapes. This, however, is eliminated in the following theorem:

Courant - Fischer's minimax principle:

$$\omega_k^2 = \max_{\dim Y = k} \left(\min_{\bar{0} \neq \bar{x} \in Y} R(\bar{x}) \right) = \min_{\dim Y = n-k+1} \left(\max_{\bar{0} \neq \bar{x} \in Y} R(\bar{x}) \right) \quad (13.10)$$

where $Y \subseteq \mathbb{R}^n$ are linear subspaces.

The proof of this is omitted.

Note that a linear space Y with $\dim Y = n-k$ may be defined by k linear constraints:

$$\bar{a}_1^T \bar{x} = 0, \bar{a}_2^T \bar{x} = 0, \dots, \bar{a}_k^T \bar{x} = 0$$

where $\bar{a}_1, \dots, \bar{a}_k$ are linearly independent vectors. This may also be written

$$\underline{A} \bar{x} = \bar{0}, \quad \underline{A} = [\bar{a}_1 \dots \bar{a}_k]$$

14. Constrained systems

We consider a mechanical system

$$S: \underline{M}, \underline{K}, \bar{q} \in \mathbb{R}^n$$

Now we impose a linear (time independent) constraint on the coordinates defined by

$$\bar{a}^T \bar{q} = 0, \quad \forall t \quad (14.1)$$

where $\bar{a}$ is a constant vector $\neq \bar{0}$.

Looking at free motion:

$$\bar{q} = \bar{x} \neq (t)$$

We find that

$$\bar{a}^T \bar{x} = 0$$

Now the modified (constrained) system

$$\tilde{S}: \underline{M}, \underline{K}, \bar{a}^T \bar{q} = 0, \bar{q} \in \mathbb{R}^n$$

is $n-1$ dimensional if S is n -dimensional.

$$\mathbb{R}^{n-1} = \{ \bar{q} \in \mathbb{R}^n : \bar{a}^T \bar{q} = 0 \} \quad (14.2)$$

We may now write:

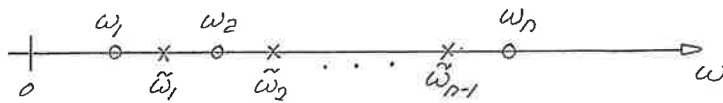
$$\tilde{S} : \tilde{M}, \tilde{K}, \tilde{q} \in \mathbb{R}^{n-1}$$

where $\tilde{M}$ and $\tilde{K}$ are the mass- and stiffness matrices for the constrained system. Let $\omega_1^2 \leq \omega_2^2 \leq \dots \leq \omega_n^2$ be the eigenfrequencies for the original system S and,

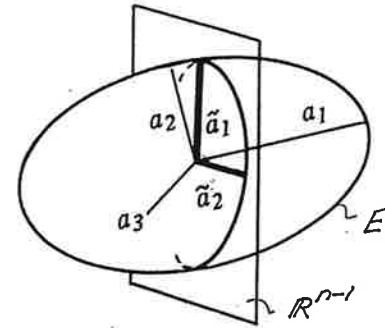
$$\tilde{\omega}_1^2 \leq \tilde{\omega}_2^2 \leq \dots \leq \tilde{\omega}_{n-1}^2 \quad (14.3)$$

the $(n-1)$ eigenfrequencies for the constrained system. Then we have the Separation theorem:

$$\omega_1^2 \leq \tilde{\omega}_1^2 \leq \omega_2^2 \leq \tilde{\omega}_2^2 \leq \dots \leq \omega_{n-1}^2 \leq \tilde{\omega}_{n-1}^2 \leq \omega_n^2 \quad (14.4)$$



The theorem follows from a consideration of the intersection of the ellipsoide for S and a plane in $\mathbb{R}^n$.



With m linearly independent constraints:

$$\bar{a}_j^T \bar{x} = 0 \quad j=1, \dots, m \quad (14.5)$$

the modified system has $n-m$ eigenfrequencies $\tilde{\omega}_i$ $i=1, \dots, n-m$ and

$$\omega_i^2 \leq \tilde{\omega}_i^2 \leq \omega_{i+m}^2 \quad (14.6)$$

The equations of motion for the constrained system may be written

138 b)

$$\begin{aligned} \underline{M} \ddot{\underline{q}} + \underline{K} \underline{q} &= \underline{\bar{R}}^c \\ \underline{\bar{a}}^T \underline{\dot{q}} &= 0 \end{aligned} \quad (14.7)$$

where $\underline{\bar{R}}^c$ represents the generalized force due to constraint forces and moments.

We assume that

$$(\underline{\bar{R}}^c)^T \underline{\bar{w}} = 0 \quad (14.8)$$

for all $\underline{\bar{w}}$ satisfying

$$\underline{\bar{a}}^T \underline{\bar{w}} = 0 \quad (14.9)$$

which means that the virtual power expended by $\underline{\bar{R}}^c$ on a virtual velocity $\underline{\bar{w}}$, satisfying the constraint condition (14.9), is equal to zero.

From (14.8) - (14.9) it follows that

$$\underline{\bar{R}}^c = \lambda \underline{\bar{a}}, \quad \lambda \in \mathbb{R} \quad (14.10)$$

Thus (14.7) may be written

$$\begin{aligned} \underline{M} \ddot{\underline{q}} + \underline{K} \underline{q} &= \lambda \underline{\bar{a}} \\ \underline{\bar{a}}^T \underline{\dot{q}} &= 0 \end{aligned} \quad (14.11)$$

138 c)

A change to normal coordinates gives

$$\begin{aligned} \underline{\mu} \ddot{\underline{\eta}} + \underline{\kappa} \underline{\eta} &= \lambda \underline{\bar{b}} \\ \underline{\bar{b}}^T \underline{\dot{\eta}} &= 0 \end{aligned} \quad (14.12)$$

where $\underline{\bar{b}} = \underline{X}^T \underline{\bar{a}}$. Componentwise we have

$$\mu_i \ddot{\eta}_i + \kappa_i \eta_i = \lambda b_i \quad (14.13)$$

Assume that

$$\eta_i = \eta_i(t) = \hat{\eta}_i \sin \tilde{\omega} t, \quad \hat{\eta}_i, \tilde{\omega} \in \mathbb{R} \quad (14.14)$$

where $\tilde{\omega}$ is the eigenfrequency of the modified system. (14.14) inserted into (14.13) gives

$$(-\mu_i \tilde{\omega}^2 + \kappa_i) \hat{\eta}_i \sin \tilde{\omega} t = \lambda b_i \Rightarrow$$

$$\lambda = \lambda(t) = \frac{\omega_i^2 - \tilde{\omega}^2}{b_i} \mu_i \hat{\eta}_i \sin \tilde{\omega} t \quad (14.15)$$

if

$$b_i \neq 0 \quad (14.16)$$

138 d)

If $b_i = 0$ then $\tilde{\omega} = \omega_i^2$. Here

$$\omega_i^2 = \frac{\kappa_i}{\mu_i} \quad (14.17)$$

is the eigenfrequency of the original system. With

$$\lambda(t) = \hat{\lambda} \sin \tilde{\omega} t, \quad \hat{\lambda} \in \mathbb{R} \quad (14.18)$$

we conclude that

$$\hat{\gamma}_i = \frac{b_i \hat{\lambda}}{(\omega_i^2 - \tilde{\omega}^2) \mu_i} \quad (14.19)$$

and from the constraint condition (14.12)₂

$$\sum_{i=1}^n \frac{b_i^2 \hat{\lambda}}{(\omega_i^2 - \tilde{\omega}^2) \mu_i} = 0 \quad (14.20)$$

Assume that ($m \leq n$)

$$b_i \neq 0, \quad i = 1, 2, \dots, m \quad (14.21)$$

$$b_i = 0, \quad i = m+1, m+2, \dots, n$$

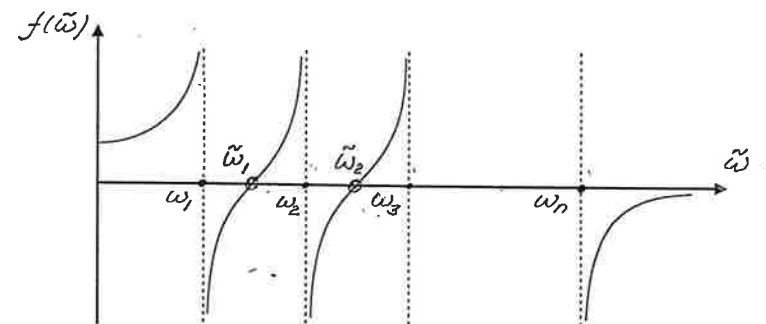
Then

$$\tilde{\omega}_i = \omega_i, \quad i = m+1, \dots, n \quad (14.22)$$

138 e)

The remaining modified angular velocities are determined by the equation

$$f(\tilde{\omega}) = \sum_{i=1}^m \frac{b_i^2}{(\omega_i^2 - \tilde{\omega}^2) \mu_i} = 0 \quad (14.23)$$



LECTURE 8

DAMPED VIBRATIONS

Diagonalization with normal modes
Rayleigh damping

DAMPED VIBRATIONS*n*-degree - of - freedom - systems15. Non - gyroscopic systems

We here assume that

$$\underline{B} = \underline{C} + \underline{G} \quad (15.1)$$

$$\underline{C}^T = \underline{C} \text{ pos. definite, } \underline{G} = \underline{0}$$

We then have the vibration equation:

$$\underline{M} \ddot{\bar{q}} + \underline{C} \dot{\bar{q}} + \underline{K} \bar{q} = \bar{P} \quad (15.2)$$

$$\bar{q}(0) = \bar{q}_0, \quad \dot{\bar{q}}(0) = \dot{\bar{q}}_0$$

$$\underline{M}^T = \underline{M}; \text{ strictly positive definite}$$

$$\underline{K}^T = \underline{K}; \text{ positive definite (semi-)}$$

The associated undamped problem is given by:

$$\underline{M} \ddot{\bar{q}} + \underline{K} \bar{q} = \bar{P} \quad (15.3)$$

and the corresponding free vibrations:

$$\underline{M}\ddot{\underline{q}} + \underline{K}\underline{q} = \underline{0}$$

We know that there exists $\omega_1^2, \omega_2^2, \dots, \omega_n^2$ and $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n$ such that:

$$(-\omega_i^2 \underline{M} + \underline{K})\bar{x}_i = \underline{0}$$

The modal matrix $\underline{X} = (\bar{x}_1 \dots \bar{x}_n)$ may be used to simultaneously diagonalize $\underline{M}$ and $\underline{K}$. With

$$\bar{q} = \underline{X}\bar{\eta}$$

we get

$$\underline{\mu}\ddot{\bar{\eta}} + \underline{\kappa}\bar{\eta} = \underline{X}^T \underline{P}$$

where

$$\underline{\mu} = \underline{X}^T \underline{M} \underline{X} = \text{diag}(\mu_1, \dots, \mu_n)$$

$$\underline{\kappa} = \underline{X}^T \underline{K} \underline{X} = \text{diag}(\kappa_1, \dots, \kappa_n)$$

$$\underline{\omega}^2 = \underline{\mu}^{-1} \underline{\kappa} = \text{diag}(\omega_1^2, \dots, \omega_n^2) = \underline{X}^{-1} \underline{M}^{-1} \underline{K} \underline{X}$$

Now if we look at the damped problem we may ask if there is a coordinate transformation:

$$\bar{q} = \underline{S}\bar{\eta}, \quad \det \underline{S} \neq 0$$

such that we simultaneously diagonalize all three matrices $\underline{M}$, $\underline{C}$ and $\underline{K}$ and then:

$$\underline{\mu}\ddot{\bar{\eta}} + \underline{\gamma}\dot{\bar{\eta}} + \underline{\kappa}\bar{\eta} = \underline{P}_S \quad (15.4)$$

$$\underline{\mu} = \underline{S}^T \underline{M} \underline{S} = \text{diag}(\mu_1, \dots, \mu_n)$$

$$\underline{\gamma} = \underline{S}^T \underline{C} \underline{S} = \text{diag}(\gamma_1, \dots, \gamma_n) \quad (15.5)$$

$$\underline{\kappa} = \underline{S}^T \underline{K} \underline{S} = \text{diag}(\kappa_1, \dots, \kappa_n)$$

The answer to this is no! For example if we take $\underline{S} = \underline{X}$ we may diagonalize $\underline{M}$ and $\underline{K}$ but nothing guarantees that:

$$\underline{\gamma} = \underline{X}^T \underline{C} \underline{X} = [\gamma_{ij}]$$

is diagonal. The equations of motion may be written:

$$\ddot{\eta}_i + \sum_{j=1}^n \frac{\delta_{ij}}{\mu_i} \dot{\eta}_j + \omega_i^2 \eta_i = \pi_i \quad i=1, \dots, n \quad (15.6)$$

$$\delta_{ij} = \bar{x}_i^T \underline{C} \bar{x}_j, \quad \pi_i = \bar{x}_i^T \bar{p}$$

The equations are thus, in general, not uncoupled. This will lead to more complicated motions of the system.

Since $\underline{M}$ is symmetric and positive definite there is a matrix $\underline{m}$ such that $\underline{m}^2 = \underline{M}$. We write $\underline{m} = \underline{M}^{\frac{1}{2}}$.

We now introduce the change of coordinates:

$$\underline{y} = \underline{M}^{\frac{1}{2}} \bar{q} \quad \Leftrightarrow \quad \bar{q} = \underline{M}^{-\frac{1}{2}} \underline{y}$$

Then equation (15.2) transforms into

$$\ddot{\underline{y}} + \underbrace{\underline{M}^{-\frac{1}{2}} \underline{C} \underline{M}^{-\frac{1}{2}}}_{=\underline{A}} \dot{\underline{y}} + \underbrace{\underline{M}^{-\frac{1}{2}} \underline{K} \underline{M}^{-\frac{1}{2}}}_{=\underline{B}} \underline{y} = \underline{M}^{-\frac{1}{2}} \bar{p}$$

or

$$\ddot{\underline{y}} + \underline{A} \dot{\underline{y}} + \underline{B} \underline{y} = \underline{M}^{-\frac{1}{2}} \bar{p} \quad (15.7)$$

where

$$\underline{A} = \underline{M}^{-\frac{1}{2}} \underline{C} \underline{M}^{-\frac{1}{2}} \Rightarrow \underline{A}^T = \underline{A} \quad (15.8)$$

$$\underline{B} = \underline{M}^{-\frac{1}{2}} \underline{K} \underline{M}^{-\frac{1}{2}} \Rightarrow \underline{B}^T = \underline{B}$$

We now introduce another change of coordinates ($\underline{y} \rightarrow \underline{z}$):

$$\underline{y} = \underline{Q} \underline{z}, \quad \det \underline{Q} \neq 0$$

Then

$$\underline{Q}^T \underline{Q} \ddot{\underline{z}} + \underline{Q}^T \underline{A} \underline{Q} \dot{\underline{z}} + \underline{Q}^T \underline{B} \underline{Q} \underline{z} = \underline{Q}^T \underline{M}^{-\frac{1}{2}} \bar{p} \quad (15.9)$$

We now would like to find $\underline{Q}$ such that:

$$\underline{Q}^T \underline{A} \underline{Q} = \text{diag}(a_1, \dots, a_n) = \underline{a}$$

$$\underline{Q}^T \underline{B} \underline{Q} = \text{diag}(b_1, \dots, b_n) = \underline{b} \quad (15.10)$$

$$\underline{Q}^T \underline{Q} = \underline{1} = \text{diag}(1, \dots, 1)$$

When is this possible?

Not always, because we have the necessary (and sufficient) condition

$$\| \underline{A}\underline{B} = \underline{B}\underline{A} \quad (15.11)$$

i.e. the matrices $\underline{A}$ and $\underline{B}$ have to commute. The matrix $\underline{Q}$ is orthogonal.

$$\underline{A}\underline{B} = \underline{B}\underline{A} \Leftrightarrow \underline{M}^{-\frac{1}{2}} \underline{C} \underline{M}^{-\frac{1}{2}} \underline{M}^{-\frac{1}{2}} \underline{K} \underline{M}^{-\frac{1}{2}} = \underline{M}^{-\frac{1}{2}} \underline{K} \underline{M}^{-\frac{1}{2}} \underline{M}^{-\frac{1}{2}} \underline{C} \underline{M}^{-\frac{1}{2}} \Leftrightarrow$$

$$\| \underline{C} \underline{M}^{-1} \underline{K} = \underline{K} \underline{M}^{-1} \underline{C} \quad (15.12)$$

i.e. $\underline{C}$ and $\underline{K}$ commute via the inverse mass matrix. This condition will be a test whether it is possible to write the vibration equation on the form:

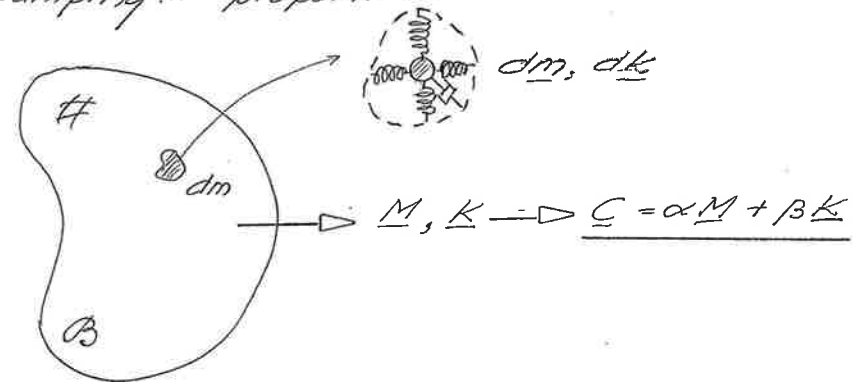
$$\| \ddot{\underline{x}} + \underline{a} \dot{\underline{x}} + \underline{b} \underline{x} = \underline{Q}^T \underline{M}^{-\frac{1}{2}} \underline{P} \quad (15.13)$$

with uncoupled component equations.

A simple damping model is given by:

$$\| \underline{C} = \alpha \underline{M} + \beta \underline{K} \quad (15.14)$$

where $\alpha, \beta \in \mathbb{R}$. This is called proportional damping or Rayleigh damping. It means that the amount of damping is a sum of two parts, one proportional to mass and the other proportional to stiffness. "Where there is mass or stiffness there is damping in proportion!"



Does the Rayleigh damping satisfy the condition (15.12)? A simple

calculation confirms that the answer is yes! We may then uncouple the equations of motion. In fact we may use the classical modal matrix to do that, since a change of coordinates ($\bar{q} \rightarrow \bar{\eta}$):

$$\bar{q} = \underline{X} \bar{\eta}$$

results in the equations:

$$\underbrace{\underline{X}^T \underline{M} \underline{X}}_{\underline{\mu}} \ddot{\bar{\eta}} + \underbrace{\underline{X}^T \underline{C} \underline{X}}_{\underline{\gamma}} \dot{\bar{\eta}} + \underbrace{\underline{X}^T \underline{K} \underline{X}}_{\underline{\kappa}} \bar{\eta} = \underbrace{\underline{X}^T \underline{D}}_{\underline{\pi}} \quad (15.15)$$

where

$$\underline{\gamma} = \underline{X}^T \underline{C} \underline{X} = \underline{X}^T (\alpha \underline{M} + \beta \underline{K}) \underline{X} =$$

$$\alpha \underline{X}^T \underline{M} \underline{X} + \beta \underline{X}^T \underline{K} \underline{X} = \alpha \underline{\mu} + \beta \underline{\kappa} \quad (15.16)$$

and thus

$$\underline{\gamma} = \text{diag}(\gamma_1, \dots, \gamma_n)$$

$$i = 1, \dots, n$$

$$\gamma_i = \alpha \mu_i + \beta \kappa_i$$

$$(15.17)$$

Eq. (15.15) is then equivalent to:

$$\begin{aligned} \mu_1 \ddot{\eta}_1 + \gamma_1 \dot{\eta}_1 + \kappa_1 \eta_1 &= \pi_1 \\ &\vdots \\ \mu_n \ddot{\eta}_n + \gamma_n \dot{\eta}_n + \kappa_n \eta_n &= \pi_n \end{aligned} \quad (15.18)$$

where:

$$\mu_i > 0, \quad \gamma_i \geq 0, \quad \kappa_i \geq 0 \quad (15.19)$$

We have thus reduced the problem (15.2) to the (one-dimensional) problem of solving the equation of type:

$$\mu \ddot{\eta} + \gamma \dot{\eta} + \kappa \eta = \pi \quad (15.20)$$

where

$$\mu > 0, \quad \gamma \geq 0, \quad \kappa \geq 0 \quad (15.21)$$

and with initial data:

$$\eta(0) = \eta_0, \quad \dot{\eta}(0) = \dot{\eta}_0 \quad (15.22)$$

A general solution to (15.20)

may be written

$$\eta = \eta_p + \eta_h \quad (15.23)$$

where η_p is a (particular) solution to (15.20) and η_h is a solution to the homogeneous equation:

$$\mu \ddot{\eta} + \gamma \dot{\eta} + \kappa \eta = 0 \quad (15.24)$$

To find the solutions to the homogeneous equation we take ("a trick")

$$\eta = \rho e^{st} \quad (\rho = \eta e^{-st}), \quad s = \text{constant} \quad (15.25)$$

Then ($\eta = \eta(t)$):

$$\dot{\eta} = \dot{\rho} e^{st} + \rho s e^{st}$$

$$\ddot{\eta} = \ddot{\rho} e^{st} + \dot{\rho} s e^{st} + \rho s^2 e^{st} + \dot{\rho} s e^{st}$$

$$\mu \ddot{\eta} + \gamma \dot{\eta} + \kappa \eta = (\mu \ddot{\rho} + (\gamma + 2\mu s)\dot{\rho} +$$

$$(\mu s^2 + \gamma s + \kappa)\rho) e^{st} = 0 \quad \forall t$$

$$\Rightarrow \mu \ddot{\rho} + (\gamma + 2\mu s)\dot{\rho} + (\mu s^2 + \gamma s + \kappa)\rho = 0 \quad (15.26)$$

Now the constant s is at our disposal.

We choose s such that:

$$\mu s^2 + \gamma s + \kappa = 0 \quad (15.27)$$

that is ($i^2 = -1$)

$$s = \begin{cases} -\frac{\gamma}{2\mu} \pm \sqrt{\left(\frac{\gamma}{2\mu}\right)^2 - \frac{\kappa}{\mu}} & \text{if } \left(\frac{\gamma}{2\mu}\right)^2 \geq \frac{\kappa}{\mu} \\ -\frac{\gamma}{2\mu} \pm i\sqrt{\frac{\kappa}{\mu} - \left(\frac{\gamma}{2\mu}\right)^2} & \text{if } \left(\frac{\gamma}{2\mu}\right)^2 < \frac{\kappa}{\mu} \end{cases} \quad (15.28)$$

then (15.26) is equivalent to

$$\mu \ddot{\rho} + (\gamma + 2\mu s)\dot{\rho} = \frac{d}{dt}(\mu \dot{\rho} + \Delta \rho) = 0 \quad (15.29)$$

where

$$\Delta = \begin{cases} \pm \sqrt{\gamma^2 - 4\mu\kappa} & \text{if } \left(\frac{\gamma}{2\mu}\right)^2 \geq \frac{\kappa}{\mu} \\ \pm i\sqrt{4\mu\kappa - \gamma^2} & \text{if } \left(\frac{\gamma}{2\mu}\right)^2 < \frac{\kappa}{\mu} \end{cases} = 2\mu s + \gamma \quad (15.30)$$

$$(15.29) \Rightarrow \mu \dot{\rho} + \Delta \rho = K_1 = \text{constant}$$

$$\Rightarrow \dot{\rho} + \frac{\Delta}{\mu} \rho = \frac{K_1}{\mu} \quad (15.31)$$

We now introduce the relative damping ratio:

$$\zeta = \frac{\gamma}{2\sqrt{k\mu}} = \frac{\gamma}{2\omega_0\mu}, \quad \omega_0^2 = \frac{\kappa}{\mu} \quad (15.32)$$

then (15.28) may be written

$$s = \begin{cases} (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_0, & \zeta \geq 1 \\ (-\zeta \pm i\sqrt{1 - \zeta^2})\omega_0, & \zeta < 1 \end{cases} \quad (15.33)$$

$\zeta > 1$: strong damping (overdamped)

$\zeta = 1$: critical damping

$\zeta < 1$: weak damping (underdamped)

and

$$\Delta = \begin{cases} \pm \sqrt{\zeta^2 - 1} \, 2\mu\omega_0 & \zeta \geq 1 \\ \pm i\sqrt{1 - \zeta^2} \, 2\mu\omega_0 & \zeta < 1 \end{cases} \quad (15.34)$$

Now if we look at (15.30) and assume $\Delta \neq 0$ (non-critical), i.e. $\zeta \neq 1$

We may multiply (15.30) with the "integrating factor" $e^{\frac{\Delta}{\mu}t}$:

$$e^{\frac{\Delta}{\mu}t} \dot{p} + \frac{\Delta}{\mu} e^{\frac{\Delta}{\mu}t} p = k_1 e^{\frac{\Delta}{\mu}t} \Rightarrow$$

$$\frac{d}{dt}(e^{\frac{\Delta}{\mu}t} p) = k_1 e^{\frac{\Delta}{\mu}t} \Rightarrow e^{\frac{\Delta}{\mu}t} p =$$

$$\frac{k_1 \mu}{\Delta} e^{\frac{\Delta}{\mu}t} + K_2 \Rightarrow p = \frac{k_1 \mu}{\Delta} + K_2 e^{-\frac{\Delta}{\mu}t}$$

where K_1 and K_2 are arbitrary complex constants. Then

$$y = p e^{st} = \frac{k_1 \mu}{\Delta} e^{st} + K_2 e^{-\frac{\Delta}{\mu}t} \cdot e^{st} =$$

$$\frac{k_1 \mu}{\Delta} e^{st} + K_2 e^{-(\zeta \pm i\sqrt{1-\zeta^2})t} e^{st} =$$

$$A e^{st} + B e^{-(\zeta \pm i\sqrt{1-\zeta^2})t} = A e^{(-\frac{\gamma}{2\mu} \pm \frac{\Delta}{\mu})t} +$$

$$B e^{(-\frac{\gamma}{2\mu} \mp \frac{\Delta}{\mu})t} =$$

$$e^{-\frac{\gamma}{2\mu}t} (A e^{\pm \frac{\Delta}{\mu}t} + B e^{\mp \frac{\Delta}{\mu}t})$$

where $A = \frac{k_1 \mu}{\Delta}$ and $B = k_2$ are arbitrary constants. We then have:

$\zeta > 1$: (overdamped)

$$\eta = e^{-\zeta \omega_0 t} (A e^{\sqrt{\zeta^2 - 1} \omega_0 t} + B e^{-\sqrt{\zeta^2 - 1} \omega_0 t}) \quad (15.35)$$

$\zeta < 1$: (underdamped)

$$\eta = e^{-\zeta \omega_0 t} (A e^{i \omega_d t} + B e^{-i \omega_d t}) \quad (15.36)$$

where

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2}, \quad \omega_0^2 = \frac{\kappa}{\mu} \quad (15.37)$$

is called the damped natural frequency.

The real solution corresponding to (15.36) reads:

$$\eta = e^{-\zeta \omega_0 t} (A \cos \omega_d t + B \sin \omega_d t) \quad (15.38)$$

where now $A, B \in \mathbb{R}$.

We have, in (15.36), used $A^* = B^*$ ¹⁾ and

$$e^{i \omega_d t} = \cos \omega_d t + i \sin \omega_d t$$

Finally we must look at the case

$\Delta = 0$: $\Leftrightarrow \zeta = 1$ (critical damping)

$$\text{Then } \mu \ddot{\rho} = 0 \Rightarrow \rho = A + Bt \Rightarrow$$

$$\eta = \rho e^{\zeta t} = (A + Bt) e^{-\omega_0 t} \quad (15.39)$$

Then we have the solution to vibration problem (15.2) with $\bar{P} = \bar{0}$, the homogeneous solution:

$$\bar{q}_h = \underline{X} \bar{\eta} = \sum_{i=1}^n \bar{x}_i e^{-\zeta_i \omega_{0i} t} (A_i \cos \omega_{di} t + B_i \sin \omega_{di} t) \quad (15.40)$$

(if $\zeta_i < 1 \forall i$)

$$\omega_{di} = \omega_{0i} \sqrt{1 - \zeta_i^2}, \quad i = 1, \dots, n$$

$$\omega_{0i}^2 = \frac{\kappa_i}{\mu_i} \quad (15.41)$$

¹⁾ $A = a + ib \Rightarrow A^* = a - ib$ complex conjugate

The solution of:

$$\begin{cases} \underline{M} \ddot{\bar{q}} + \underline{C} \dot{\bar{q}} + \underline{K} \bar{q} = \bar{0} \\ \bar{q}(0) = \bar{q}_0, \quad \dot{\bar{q}}(0) = \dot{\bar{q}}_0 \end{cases} \quad (15.42)$$

may now be written:

$$\begin{aligned} \bar{q} = & \left(\sum_{i=1}^n e^{-\zeta_i \omega_{di} t} \left(\frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i} \cos \omega_{di} t + \right. \right. \\ & \left. \left. \zeta_i \frac{\omega_{di}}{\omega_{di}} \frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i} \sin \omega_{di} t \right) \bar{q}_0 + \right. \\ & \left. \left(\sum_{i=1}^n e^{-\zeta_i \omega_{di} t} \frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i \omega_{di}} \sin \omega_{di} t \right) \dot{\bar{q}}_0 \right) \end{aligned} \quad (15.43)$$

Put:

$$s_i = -\zeta_i \omega_{di} \pm i \omega_{di} \quad \downarrow = \sqrt{-1} \quad i = 1, \dots, n$$

$\Rightarrow$ (of course!)

$$\mu_i s_i^2 + \gamma_i s_i + \kappa_i = 0 \quad (15.44)$$

We now introduce the matrices

$$\underline{S} = \text{diag}(s_1, \dots, s_n) = 0$$

$$\underline{S}^2 = \text{diag}(s_1^2, \dots, s_n^2)$$

$$(15.44) \Rightarrow$$

$$\underline{\mu} \underline{S}^2 + \underline{\gamma} \underline{S} + \underline{\kappa} = \underline{0} \quad \Rightarrow$$

$$\underline{X}^T \underline{M} \underline{X} \underline{S}^2 + \underline{X}^T \underline{C} \underline{X} \underline{S} + \underline{X}^T \underline{K} \underline{X} = \underline{0} \Leftrightarrow$$

$$\underline{M} \underline{X} \underline{S}^2 + \underline{C} \underline{X} \underline{S} + \underline{K} \underline{X} = \underline{0}$$

$$\Leftrightarrow (\underline{M} \underline{X} \underline{S}^2 = [\underline{M} \bar{x}_1 s_1^2 \dots \underline{M} \bar{x}_n s_n^2])$$

$$\underline{M} \bar{x}_i s_i^2 + \underline{C} \bar{x}_i s_i + \underline{K} \bar{x}_i = \bar{0}$$

$$\Leftrightarrow (\underline{M} s_i^2 + \underline{C} s_i + \underline{K}) \bar{x}_i = \bar{0}$$

$$\underline{Z}(s_i) \bar{x}_i = \bar{0}, \quad i = 1, \dots, n \quad (15.45)$$

where $\underline{Z} = \underline{Z}(s) = \underline{M} s^2 + \underline{C} s + \underline{K}$ is the impedance matrix. We have

$$\bar{x}_i \neq \bar{0} \Rightarrow \det \underline{Z}(s_i) = 0 \quad (15.46)$$

The characteristic polynomial is defined by:

$$p(s) = \det \underline{Z}(s) = s^{2n} \det \underline{M} + \dots + \det \underline{K} \quad (15.47)$$

From the previous discussion we know that:

$$p(s_i) = 0, \quad p(s_i^*) = 0 \quad (15.48)$$

$$s_i = \begin{cases} (1 - \zeta \pm \sqrt{\zeta^2 - 1}) \omega_0 & \zeta \geq 1 \\ (-\zeta \pm i\sqrt{1 - \zeta^2}) \omega_0 & \zeta < 1 \end{cases} \quad (15.49)$$

That $p(s) = 0 \Rightarrow p(s^*) = 0$ also follows from the fact that p has real coefficients. We also have:

$$\underline{Z}(s_i^*) \underline{x}_i = \underline{0} \quad (15.50)$$

Note that if $\zeta_i = 0$ ($\zeta = 0$) then

$$s = \pm i\omega_0 \quad (15.51)$$

and

$$p(\pm i\omega_0) = \det(\underline{M}(\pm i\omega_0)^2 + \underline{K}) = \det(-\omega_0^2 \underline{M} + \underline{K}) \quad (15.52)$$

and we are back to the characteristic polynomial for the undamped system.

The characteristic polynomial may be written:

$$p(s) = \det \underline{M} (s - s_1)(s - s_1^*) \dots (s - s_n)(s - s_n^*) \quad (15.53)$$

Not all s_i have to be different. We

may have so-called multiple roots

(double root, triple root etc.). If s_i

is a root with multiplicity m then

$$p(s) = (s - s_i)^m q(s) \quad (15.54)$$

where $q = q(s)$ is a polynomial (of degree = $2n - m$) with $q(s_i) \neq 0$.

We have the following criterion:

s_i is a root to the char. eq. with multiplicity m if and only if:

$$p(s_i) = p'(s_i) = \dots = p^{(m-1)}(s_i) = 0 \quad (15.55)$$

$$p^{(m)}(s_i) \neq 0$$

The roots of the characteristic equation may be arranged in the following way:

$$s_1, \dots, s_r \text{ all real } (\in \mathbb{R}) \quad (15.56)$$

$$s_{r+1}, s_{r+1}^*, \dots, s_{r+k}, s_{r+k}^* \text{ all complex } (\in \mathbb{C})$$

where $r + 2k = n$.

By using the relation:

$$(s - s_j)(s - s_j^*) = s^2 - s(s_j + s_j^*) + s_j s_j^* = s^2 - s \, 2\operatorname{Re}(s_j) + |s_j|^2 \quad (15.57)$$

we may write:

$$p(s) = \det \underline{M} (s - s_1)(s - s_2) \dots (s - s_r) (s_{r+1}^2 - s \, 2\operatorname{Re}(s_{r+1}) + |s_{r+1}|^2) \dots (s_{r+k}^2 - s \, 2\operatorname{Re}(s_{r+k}) + |s_{r+k}|^2) \quad (15.58)$$

We may now formulate the eigenvalue-problem associated with the free vibrations of a damped mechanical

system

$$\underline{M} \ddot{\underline{q}} + \underline{C} \dot{\underline{q}} + \underline{K} \underline{q} = \underline{0} \quad (15.59)$$

Eigenvalue problem:

Given a mechanical system with impedance:

$$\underline{Z}(s) = \underline{M} s^2 + \underline{C} s + \underline{K} \quad (15.60)$$

find $s \in \mathbb{C}$ and $\underline{\bar{x}} \in \mathbb{R}^n$, $\underline{\bar{x}} \neq \underline{0}$ such that:

$$\underline{Z}(s) \underline{\bar{x}} = \underline{0} \quad (15.61)$$

We know that this problem is solvable if we have Rayleigh-damping ($\underline{C} = \alpha \underline{M} + \beta \underline{K}$) and then we may choose $\underline{\bar{x}} = \underline{\bar{x}}_i$ where $\underline{\bar{x}}_i$ is a mode-shape corresponding to the undamped problem

$$\underline{M} \ddot{\underline{q}} + \underline{K} \underline{q} = \underline{0}$$

The corresponding s -value (s_1) satisfies:

$$p(s_1) = 0 \quad (15.62)$$

Note that for a general damping matrix we may have to look for complex eigenvectors $\vec{x} \in \mathbb{C}^n$. More about this later on.

LECTURE 14

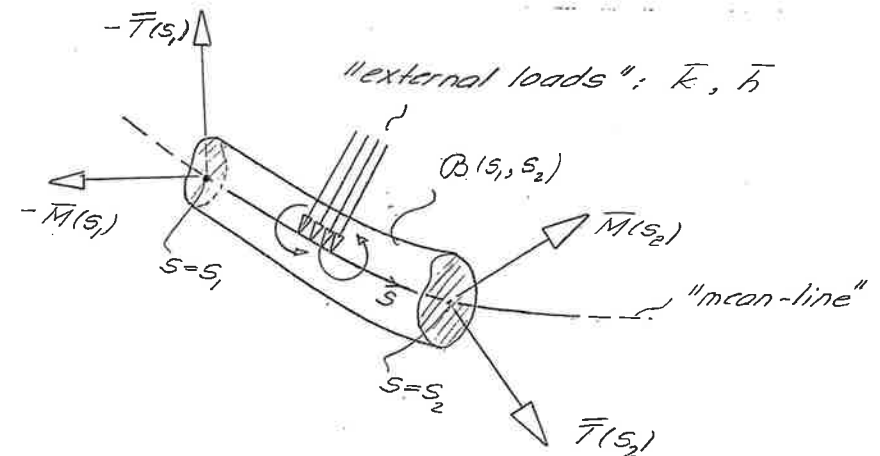
CONTINUOUS SYSTEMS

The one-dimensional body
The bar in extension

21. One-dimensional bodies

In this chapter we look at more general one-dimensional bodies. Examples of such bodies are, besides the string, the bar (rod) and the beam.

Consider a one-dimensional body B . We make a "free-body" diagram of a part of the body $B(s_1, s_2)$ according to the figure below.



The body is characterized by its "mean line" (centre-line) which is given as a space-curve:

$$\bar{F} = \bar{F}(s, t)$$

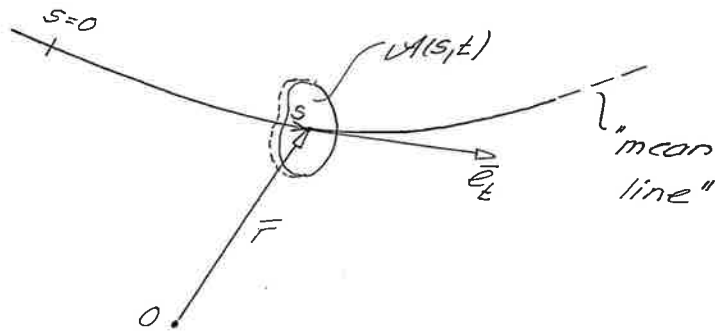
(21.1)

and the cross-sections:

$$U_A = U_A(s, t)$$

(21.2)

with $\text{area}(U_A) = A(s, t)$. It is assumed that the centroid of $U(s, t)$ is equal to $\bar{F}(s, t)$.



s denotes the arc-length coordinate.

The (unit) tangent vector of the mean-line is given by:

$$\bar{e}_t = \frac{\partial \bar{F}}{\partial s}$$

(21.3)

the internal forces "sectional quantities" acting on body segments:

The sectional force (the traction):

$$\bar{T} = \bar{T}(s, t) \quad [N]$$

(21.4)

The sectional moment:

$$\bar{M} = \bar{M}(s, t) \quad [Nm]$$

(21.5)

External forces are given by,

The external force density:

$$\bar{k} = \bar{k}(s, t) \quad [Nm^{-1}]$$

(21.6)

The external moment density:

$$\bar{h} = \bar{h}(s, t) \quad [Nm m^{-1}]$$

(21.7)

These densities are given per unit length of the "mean line" of the deformed body.

The equations of motion (according to Euler):

$$\int_{s_1}^{s_2} \bar{k} \, ds + \bar{T}(s_2) - \bar{T}(s_1) = \int_{s_1}^{s_2} \bar{a} \, m \, ds \quad (21.8)$$

$$\int_{S_1}^{S_2} (\bar{F} \times \bar{k} + \bar{h}) ds + \bar{F}(S_2) \times \bar{T}(S_2) + \bar{M}(S_2) - \bar{F}(S_1) \times \bar{T}(S_1) - \bar{M}(S_1) = \int_{S_1}^{S_2} \bar{F} \times \bar{a} m ds \quad (21.9)$$

where

$$m = m(s, t) \quad (21.10)$$

is the mass per arc-length (of the mean-line) and

$$\bar{a} = \bar{a}(s, t) \quad (21.11)$$

is the acceleration of the material point (at s). Now we put $S_2 = S$ in eq. (21.8) and (21.9) and then take the derivative of these eq. with respect to s :

$$(21.8) \Rightarrow \bar{k} + \frac{\partial \bar{T}}{\partial s} = \bar{a} m \quad (21.12)$$

$$(21.9) \Rightarrow \bar{F} \times \bar{k} + \bar{h} + \frac{\partial \bar{F}}{\partial s} \times \bar{T} +$$

$$\bar{F} \times \frac{\partial \bar{T}}{\partial s} + \frac{\partial \bar{M}}{\partial s} = \bar{F} \times \bar{a} m \quad (21.13)$$

Combining (21.12) and (21.13) we get

$$\bar{e}_t \times \bar{T} + \frac{\partial \bar{M}}{\partial s} + \bar{h} = \bar{0} \quad (21.14)$$

where $\bar{e}_t$ is the unit tangent vector to the mean-line. We thus have the following equations of motion:

$$\frac{\partial \bar{T}}{\partial s} + \bar{k} = \bar{a} m \quad (21.15)$$

$$\bar{e}_t \times \bar{T} + \frac{\partial \bar{M}}{\partial s} + \bar{h} = \bar{0}$$

If we specialize this to the vibrating string (assuming; $\bar{M} = \bar{0}$, $\bar{h} = \bar{0}$, $\bar{k} = \bar{e}_t p$ and $\bar{a} = \bar{e}_t \ddot{w}$) then we get:

$$\frac{\partial \bar{T}}{\partial s} + \bar{e}_t p = \bar{e}_t m \ddot{w} \quad (21.16)$$

$$\bar{e}_t \times \bar{T} = \bar{0} \Leftrightarrow \bar{T} = \bar{e}_t T$$

Now in this case $w = w(x, t)$, $0 \leq x \leq L$ where L is the length of the

undeformed string. We then have:

$$S = \int_0^x \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} dx = S(x, t) \quad (21.17)$$

$$\frac{\partial \bar{T}}{\partial s} = \frac{\partial \bar{T}}{\partial x} \frac{\partial x}{\partial s} \quad \left(\frac{\partial s}{\partial x} \neq 0 \right)$$

and by introducing:

$$p_0 = p \frac{\partial s}{\partial x} \quad (p_0 dx = p ds) \quad (21.18)$$

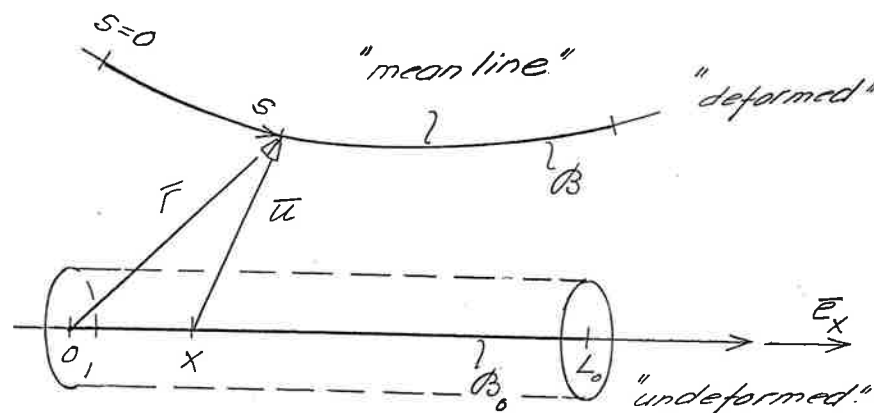
$$m_0 = m \frac{\partial s}{\partial x} \quad (m_0 dx = m ds)$$

i.e. densities measured relative to the undeformed string, we get

$$\frac{\partial \bar{T}}{\partial x} + \bar{e}_x p_0 = \bar{e}_x m_0 \ddot{u}, \quad 0 \leq x \leq L \quad (21.19)$$

which is identical to (20.5), if we identify p and p_0 , m and m_0 !

We now assume that the one-dimensional body is straight in the referential configuration, i.e. the mean-line is a straight line. See figure below!



$\bar{u}$: displacement

$$\bar{r} = \bar{e}_x x + \bar{u}(x, t), \quad 0 \leq x \leq L_0$$

$$\frac{\partial \bar{r}}{\partial x} = \bar{e}_x + \frac{\partial \bar{u}}{\partial x} \quad (21.20)$$

$$S = \int_0^x \left| \frac{\partial \bar{r}}{\partial x} \right| dx = S(x) \Leftrightarrow x = x(s)$$

The equations of motion:

$$\frac{\partial \bar{T}}{\partial x} + \bar{k}_0 = \bar{a} m_0$$

(21.21)

$$\frac{\partial \bar{T}}{\partial x} \times \bar{T} + \frac{\partial \bar{M}}{\partial x} + \bar{h}_0 = \bar{0}$$

where

$$\bar{k}_0 = \bar{k} \frac{\partial S}{\partial x}, \quad \bar{h}_0 = \bar{h} \frac{\partial S}{\partial x} \quad (21.22)$$

Now let:

$$\bar{u} = \bar{e}_x u + \bar{e}_z w \quad (\text{plane deformation})$$

$$u = u(x, t), \quad w = w(x, t) \quad (\text{displacements})$$

$$\bar{T} = \bar{e}_x N + \bar{e}_z S$$

$$N = N(x, t) : \text{"Normal (axial) force"}$$

$$S = S(x, t) : \text{"Shear force"}$$

$$\bar{k}_0 = \bar{e}_x k_0 + \bar{e}_z p_0$$

$$\bar{M} = \bar{e}_y M$$

$$M = M(x, t) : \text{"Bending moment"}$$

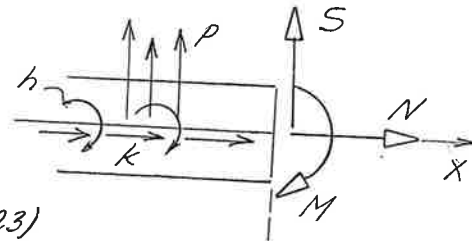
$$\bar{h}_0 = \bar{e}_y h_0$$

then (21.21)₁ is equivalent to:

$$\frac{\partial N}{\partial x} + k_0 = \ddot{u} m_0$$

$$\frac{\partial S}{\partial x} + p_0 = \ddot{w} m_0$$

(21.23)



$$(21.21)_2 \Leftrightarrow$$

$$(\bar{e}_x (1 + \frac{\partial u}{\partial x}) + \bar{e}_z \frac{\partial w}{\partial x}) \times (\bar{e}_x N + \bar{e}_z S)$$

$$+ \bar{e}_y \frac{\partial M}{\partial x} + \bar{e}_y h_0 = \bar{0} \quad \Leftrightarrow$$

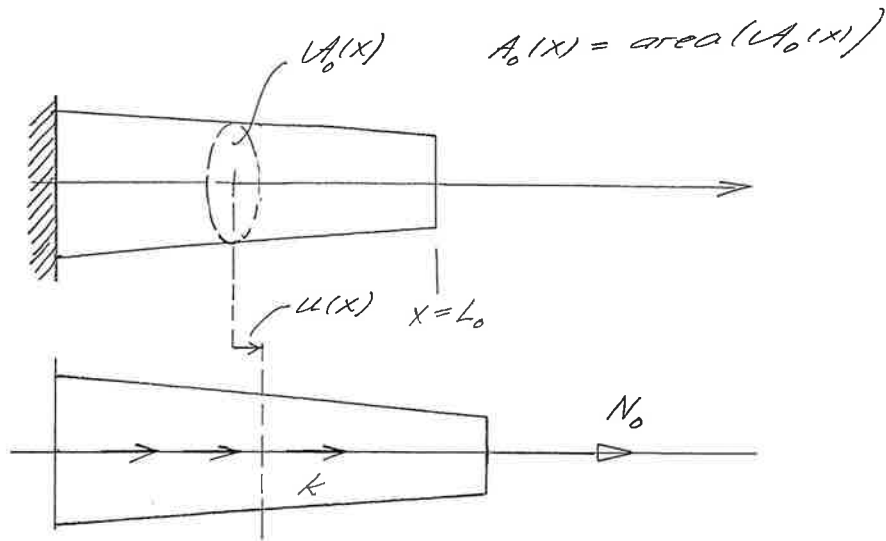
$$- S(1 + \frac{\partial u}{\partial x}) + N \frac{\partial w}{\partial x} + \frac{\partial M}{\partial x} + h_0 = 0 \quad (21.24)$$

By differentiating this equation we get:

$$- \frac{\partial S}{\partial x} (1 + \frac{\partial u}{\partial x}) - S \frac{\partial^2 u}{\partial x^2} + \frac{\partial N}{\partial x} \frac{\partial w}{\partial x} +$$

$$N \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 M}{\partial x^2} + \frac{\partial h_0}{\partial x} = 0 \quad (21.25)$$

The bar in extension



We assume that:

$$W = 0, \quad u = u(x, t)$$

$$S = 0, \quad p_0 = 0 \quad (21.26)$$

$$M = 0, \quad h_0 = 0$$

$$N = N(x, t), \quad k_0 = k_0(x, t)$$

The only non-trivial eq. is (21.23),:

$$\frac{\partial N}{\partial x} + k_0 = \ddot{u} m_0 \quad (21.27)$$

Note that eq. (21.23)₂ and (21.24) are trivially satisfied.

Now we introduce the "constitutive equation":

$$N(x, t) = E A_0(x) \frac{\partial u}{\partial x}(x, t) \quad (21.28)$$

i.e. the normal force is proportional to the "strain" $\epsilon_x = \frac{\partial u}{\partial x}$.

With (21.28) inserted into (21.27):

$$\frac{\partial}{\partial x} (E A_0 \frac{\partial u}{\partial x}) + k_0 = \ddot{u} m_0 \quad (21.29)$$

which is the governing partial differential eq. for the bar in extension! One example of boundary condition is:

$$u(0, t) = 0 \quad \text{fixed end} \quad (21.30)$$

$$N(L_0, t) = N_0(t) \quad \text{prescribed force}$$

Note that

$$N(L_0, t) = N_0(t) \Leftrightarrow E A_0(L_0) \frac{\partial u}{\partial x}(L_0, t) = N_0(t)$$

(21.31)

Free vibrations: $k_0 = 0$

We also assume that:

$$A_0 = \text{constant}$$

We look for "standing wave" solutions

$$u(x, t) = \hat{u}(x) \cos \omega t \quad (21.32)$$

This inserted into (21.29) results in

$$E A_0 \frac{d^2 \hat{u}}{dx^2} + \omega^2 m_0 \hat{u} = 0 \quad (21.33)$$

which admits the general solution:

$$\hat{u}(x) = a \sin \lambda \frac{x}{L_0} + b \cos \lambda \frac{x}{L_0} \quad (21.34)$$

where the non-dimensional parameter

$$\lambda^2 = \omega^2 m_0 L_0^2 = \frac{E A_0}{C^2} \frac{L_0^2}{L_0^2}, \quad C^2 = \frac{E A_0}{m_0} \quad (21.35)$$

For the "clamped-free" case, the boundary conditions are:

$$u(0, t) = 0, \quad \frac{\partial u}{\partial x}(L_0, t) = 0 \quad (21.36)$$

$\Leftrightarrow$

$$\left\{ \begin{array}{l} b = 0 \\ \cos \lambda = 0 \end{array} \right. \Leftrightarrow \lambda_n = -\frac{\pi}{2} n, \quad n = 1, 2, \dots$$

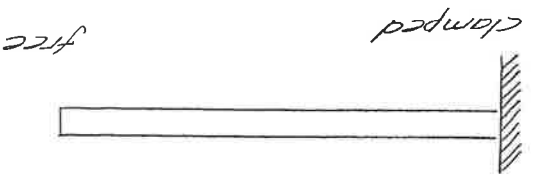
and then the corresponding natural frequencies:

$$\omega_n = \lambda_n \sqrt{\frac{E A_0}{m_0 L_0^2}} = (2n-1) \frac{\pi}{2} \sqrt{\frac{E A_0}{m_0 L_0^2}} \quad (21.37)$$

and the associated mode shapes:

$$\hat{u}_n(x) = a \sin \frac{(2n-1)\pi x}{2L_0} \quad n = 1, 2, \dots$$

where $a \in \mathbb{R}$ is an arbitrary constant, (amplitude).

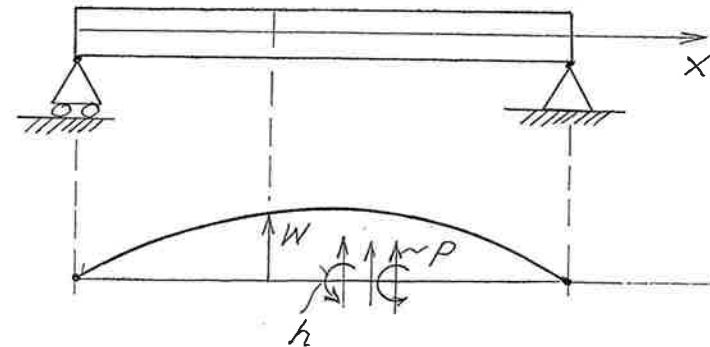


LECTURE 15

CONTINUOUS SYSTEMS

The beam in bending

The beam in bending (Euler-Bernoulli)



Simply supported beam

We assume that:

$$u = 0, \quad w = w(x, t)$$

$$N = 0, \quad k = 0 \quad (\text{no axial forces})$$

(21.39)

$$S = S(x, t), \quad p = p(x, t)$$

$$M = M(x, t), \quad h = h(x, t)$$

Eq: (21.23)₂ gives:

$$\frac{\partial S}{\partial x} + p = \ddot{w} m_0$$

(21.40)

Eq. (21.25) $\Rightarrow$

$$\left\| -\frac{\partial S}{\partial x} + \frac{\partial^2 M}{\partial x^2} + \frac{\partial h}{\partial x} = 0 \right. \quad (21.41)$$

The two equations (21.40) and (21.41) contain the unknowns;

S , W and M

We thus need an additional equation! This is supplied by the "constitutive assumption" ($M = -EI_0 \kappa$, $\kappa = \text{curvature}$)

$$\left\| M(x, t) = -EI_0(x) \underbrace{\frac{\partial^2 W}{\partial x^2}(x, t)}_{\cong \kappa} \right. \quad (21.42)$$

where

$$I_0 = I_0(x)$$

is the area moment of inertia of the section $A_0(x)$.

(21.40), (21.41) and (21.42) $\Rightarrow$

$$\left\| \frac{\partial^2}{\partial x^2} (EI_0 \frac{\partial^2 W}{\partial x^2}) + \ddot{W} m_0 = p + \frac{\partial h}{\partial x} \right. \quad (21.43)$$

Note that this is a fourth order diff. eq. (in x)!

The boundary conditions:

$$\left\| W(0, t) = W(L_0, t) = 0 \right. \quad (21.44)$$

$$\left\| M(0, t) = M(L_0, t) = 0 \right.$$

representing "no displacement" and "no moment" at the end-points of the beam. Note that (21.44)₂ is equivalent to:

$$\left\| \frac{\partial^2 W}{\partial x^2}(0, t) = \frac{\partial^2 W}{\partial x^2}(L_0, t) = 0 \right. \quad (21.45)$$

Free vibrations ($p=0$, $h=0$)

We also assume that:

$$\left\| I_0 = \text{constant} \right.$$

then

$$\left\| \frac{\partial^4 W}{\partial x^4} + \overbrace{\frac{m_0}{EI}}^{\frac{1}{C^2}} \frac{\partial^2 W}{\partial t^2} = 0 \right. \quad (21.46)$$

We look for "standing wave" solutions:

$$W(x, t) = \hat{W}(x) \sin \omega t \quad (21.47)$$

$\Rightarrow$

$$\left\| \frac{d^4 \hat{W}}{dx^4} - \omega^2 \frac{m_0}{EI_0} \hat{W} = 0 \quad (21.48) \right.$$

Introduce

$$\left\| \mu^4 = \frac{\omega^2 m_0}{EI_0} \quad (21.49) \right.$$

Then a general solution of (21.48) is given by

$$\left\| \hat{W}(x) = a \sin \mu x + b \cos \mu x + c \sinh \mu x + d \cosh \mu x \quad (21.50) \right.$$

For the "simply supported" boundary conditions (21.44) we get

$$b + d = 0$$

$$a \sin \mu L_0 + b \cos \mu L_0 + c \sinh \mu L_0 +$$

$$d \cosh \mu L_0 = 0$$

$$\mu^2(-b + d) = 0$$

$$\mu^2(-a \sin \mu L_0 - b \cos \mu L_0 + c \sinh \mu L_0 +$$

$$d \cosh \mu L_0) = 0$$

$\Rightarrow$

$$\underline{b = d = 0}$$

$$\begin{cases} a \sin \mu L_0 + c \sinh \mu L_0 = 0 \\ a \sin \mu L_0 - c \sinh \mu L_0 = 0 \end{cases}$$

$$(a, c) \neq (0, 0) \Rightarrow$$

$$\sin \mu L_0 \sinh \mu L_0 = 0 \quad (\mu \neq 0) \Leftrightarrow$$

$$\sin \mu L_0 = 0 \Leftrightarrow$$

$$\mu L_0 = n\pi, \quad n = 1, 2, \dots$$

$$\left\| \mu_n = n \frac{\pi}{L_0}, \quad n = 1, 2, \dots \quad (21.51) \right.$$

$\Rightarrow \underline{c = 0}$ and the natural frequencies.

$$\omega_n = \mu_n^2 \sqrt{\frac{EI_0}{m_0}} = \frac{n^2 \pi^2}{L_0^2} \sqrt{\frac{EI_0}{m_0}}, \quad n = 1, 2, \dots \quad (21.52)$$

and the associated mode shapes:

$$\hat{W}_n(x) = a \sin \frac{n\pi x}{L_0} \quad n = 1, 2, \dots \quad (21.53)$$

Ex 16:

calculate the response to the external excitation according to the figure below: ($h \equiv 0$)

$$p = p(x, t) = p_0 \frac{x}{L_0}, \quad 0 \leq x \leq L_0, \quad t \geq 0 \quad (21.54)$$

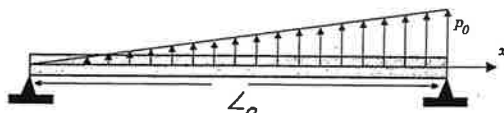


Figure 4.5.14
Excitation applied to
a simply supported beam

We assume that the beam is at rest when the load is applied.

$$w(x, 0) = 0, \quad \dot{w}(x, 0) = 0 \quad (21.55)$$

We make the following "ansatz"
(mode decomposition)

$$W(x, t) = \sum_{i=1}^{\infty} \hat{W}_i(x) \eta_i(t) \quad (21.56)$$

where $\hat{W}_i(x) = \sin \frac{i\pi x}{L_0}$, "mode shape"

$$\eta_i = \eta_i(t)$$

are "the normal coordinates" to be determined. Inserted into the diff. eq.

$$EI_0 \frac{\partial^4 W}{\partial x^4} + m_0 \frac{\partial^2 W}{\partial t^2} = p \quad (21.57)$$

This gives

$$\sum_{i=1}^{\infty} EI_0 \frac{d^4 \hat{W}_i}{dx^4} \eta_i + \sum_{i=1}^{\infty} m_0 \hat{W}_i \ddot{\eta}_i = p \quad (21.58)$$

But

$$\frac{d^4 \hat{W}_i}{dx^4} = \mu_i^4 \hat{W}_i \quad (21.59)$$

and then

$$\sum_{i=1}^{\infty} (\underbrace{EI_0 \mu_i^4}_{=\omega_i^2 m_0} \eta_i + m_0 \ddot{\eta}_i) \hat{W}_i = p \quad (21.60)$$

i.e.

$$\sum_{i=1}^{\infty} (\omega_i^2 \eta_i + \ddot{\eta}_i) \hat{W}_i = \frac{p}{m_0} \quad (21.61)$$

But the modes $\hat{W}_i$ are orthogonal

$$\int_0^{L_0} \hat{W}_i(x) \hat{W}_j(x) dx = \begin{cases} 0 & i \neq j \\ \frac{L_0}{2} & i = j \end{cases} \quad (21.62)$$

and then (21.61) is equivalent to: ($i=1, \dots, n$)

$$(\omega_i^2 \eta_i + \ddot{\eta}_i) \frac{L_0}{2} = \frac{1}{m_0} \int_0^{L_0} \hat{W}_i(x) p(x) dx \quad (21.63)$$

$$\begin{aligned} \ddot{\eta}_i + \omega_i^2 \eta_i &= \frac{2}{m_0 L_0} \int_0^{L_0} \sin \frac{i\pi x}{L_0} p_0 \frac{x}{L_0} dx \\ &= \frac{2p_0}{i\pi m_0} (-1)^{i+1} \end{aligned} \quad (21.64)$$

Initial conditions:

$$\eta_i(0) = 0, \quad \dot{\eta}_i(0) = 0 \quad (21.65)$$

=0

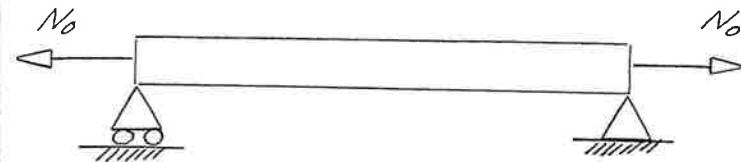
$$\eta_i(t) = \frac{2p_0(-1)^{i+1}}{i\pi m_0 \omega_i^2} (1 - \cos \omega_i t) \quad (21.66)$$

and finally:

$$w(x,t) = \sum_{i=1}^{\infty} \frac{2p_0(-1)^{i+1}}{i\pi m_0 \omega_i^2} (1 - \cos \omega_i t) \sin \frac{i\pi x}{L_0} \quad (21.67)$$

#

Pre-stressed beam in bending



N_0 : axial pre-stress

This gives:

$$N(x) = N_0, \quad 0 \leq x \leq L_0 \quad (21.68)$$

and from (21.85) we get

$$-\frac{\partial^5}{\partial x^5} + N_0 \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 M}{\partial x^2} + \frac{\partial^2}{\partial x^2} = 0 \quad (21.69)$$

and we get the equation:

$$EI_0 \frac{\partial^4 W}{\partial x^4} - N_0 \frac{\partial^2 W}{\partial x^2} + W M_0 = P + \frac{\partial^2}{\partial x^2} \quad (21.70)$$

considering the case of free vibrations (p=0, h=0) of the simply supported beam and standing wave solutions:

$$W_n(x) = a \sin \frac{n\pi x}{L_0} \quad n = 1, 2, \dots \quad (21.71)$$

This function, once again, satisfies the boundary conditions. Substitution into:

$$EI_0 \frac{\partial^4 W}{\partial x^4} - N_0 \frac{\partial^2 W}{\partial x^2} - \omega^2 M_0 W = 0 \quad (21.72)$$

gives the condition:

$$EI_0 \left(\frac{n\pi}{L_0} \right)^4 + N_0 \left(\frac{n\pi}{L_0} \right)^2 - M_0 \omega_n^2 = 0$$

=>

$$\omega_n^2 = \left(\frac{n\pi}{L_0} \right)^4 \left(\frac{EI_0}{M_0} \left(1 - \frac{N_0}{N_{crn}} \right) \right) \quad (21.73)$$

where

$$N_{crn} = - \frac{M_0 \pi^2 EI_0}{L_0^2} \quad (21.74)$$

obviously:

$$N_0 > 0 \Rightarrow \omega_n > \omega_n > \sqrt{\frac{M_0 \pi^2 EI_0}{L_0^2}} \quad (21.75)$$

(compare with (21.52))

$$- \frac{M_0 \pi^2 EI_0}{L_0^2} = N_{cr1} < N_0 < 0 \Rightarrow$$

$$\omega_n < \sqrt{\frac{M_0 \pi^2 EI_0}{L_0^2}} \quad (21.76)$$

$N_0 \rightarrow N_{cr1} = 0$ instability (Euler)

buckling.)

Free vibrations of rotating beam

(propeller, helicopter, turbine blades)

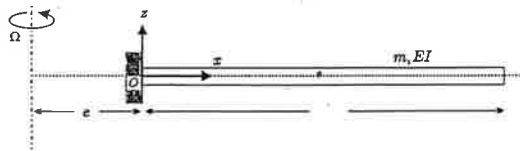


Figure 4.5.16
Vibrations of a rotating beam

The rotation gives rise to an axial prestress of the body. We have

$$k = m_0 \Omega^2 (e + x + u) \quad (21.77)$$

where $u = u(x, t)$ is the material displacement in the x -direction.

We then have:

$$\frac{dN}{dx} + m_0 \Omega^2 (e + x + u) = \ddot{u} m_0 \quad (21.78)$$

$$\frac{\partial S}{\partial x} = \ddot{u} m_0 \quad (p=0) \quad (21.79)$$

$$- \frac{\partial S}{\partial x} \left(1 + \frac{\partial u}{\partial x}\right) - S \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x} \left(N \frac{\partial u}{\partial x}\right) +$$

$$\frac{\partial^2 M}{\partial x^2} = 0 \quad (h=0) \quad (21.80)$$

With constitutive assumptions:

$$N = EA \frac{\partial u}{\partial x}, \quad M = -EI \frac{\partial^2 u}{\partial x^2} \quad (21.81)$$

Thus we have five eq. and five unknowns: N, u, S, W and M .

We now make the simplification

$$u \equiv 0$$

Then

$$\frac{\partial N}{\partial x} + m_0 \Omega^2 (e + x) = 0 \quad \Rightarrow$$

$$N = - \int_{L_0}^x m_0 \Omega^2 (e + \xi) d\xi + N(L_0)$$

$$N(L_0) = 0 \quad \Rightarrow$$

$$N(x) = m_0 \Omega^2 \left(\frac{(L_0 + e)^2}{2} - \frac{(x + e)^2}{2} \right) \quad (21.82)$$

and we then get:

$$\left| \frac{\partial^2}{\partial x^2} \left(EI_0 \frac{\partial^2 W}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(m_0 \Omega^2 \left(\frac{L_0 + e}{2} - \frac{(x+e)^2}{2} \right) \frac{dW}{dx} \right) + \ddot{W} m_0 = 0 \right. \quad (21.83)$$

Try to figure out the relevant boundary conditions!

We now take a further look at:

The beam in bending. Revisited.
(Timoshenko)

Two things are incorporated:

- rotational inertia of beam segments.
- shearing deformation.

Rotational inertia is added to the general equation of motion for an one-dimensional body by replacing the right hand side in (21.9) with

$$\left\| \int_{S_1}^{S_2} (\bar{\mathbf{r}} \times \bar{\alpha} m + \underline{\mathbf{I}} \bar{\alpha}) ds \right. \quad (21.84)$$

where $\underline{\mathbf{I}}$ is the inertia tensor and $\bar{\alpha}$ the angular acceleration of the beam segment.

The local equation (21.15)₂ is then

changed into:

$$\underline{\tilde{e}}_z \times \underline{\tilde{r}} + \frac{\partial \underline{\tilde{M}}}{\partial s} + \underline{\tilde{h}} = \underline{\tilde{I}} \underline{\tilde{\alpha}} \quad (21.85)$$

For the straight beam deforming in the plane we take:

$$\underline{\tilde{I}} \underline{\tilde{\alpha}} = \underline{\tilde{e}}_y \underline{\tilde{I}}_0 \alpha \quad (21.86)$$

and then the eq. corresponding to (21.24) reads

$$-S(1 + \frac{\partial u}{\partial x}) + N \frac{\partial w}{\partial x} + \frac{\partial M}{\partial x} + h = \underline{\tilde{I}}_0 \alpha \quad (21.87)$$

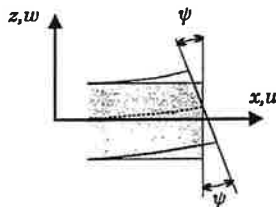
where

$$\underline{\tilde{I}}_0 = m_0 r^2, \quad r^2 = \frac{1}{A_0} \int \tilde{z}^2 da, \quad A_0 = \int da \quad (21.88)$$

r = gyration radius of beam section.

$$\alpha = -\ddot{\psi} \quad (21.89)$$

ψ is the rotation of the beam section, "the shearing angle".



We thus have the equations:

$$\frac{\partial N}{\partial x} + k = \ddot{u} m_0$$

$$\frac{\partial S}{\partial x} + p = \ddot{w} m_0 \quad (21.90)$$

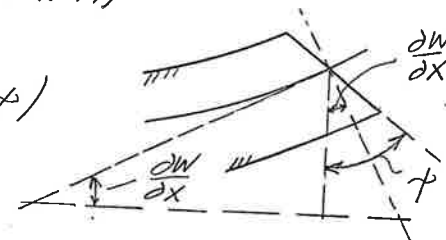
$$-S(1 + \frac{\partial u}{\partial x}) + N \frac{\partial w}{\partial x} + \frac{\partial M}{\partial x} + h = -m_0 r^2 \ddot{\psi}$$

Constitutive assumptions:

$$N = EA_0 \frac{\partial u}{\partial x} \quad (21.91)$$

$$S = \beta A_0 G (\frac{\partial w}{\partial x} - \psi)$$

$$M = -EI_0 \frac{\partial \psi}{\partial x}$$



and we end up with six equations and six unknowns: u, w, ψ, N, S, M !

Introduced constants:

A_0 : section area

G : shear modulus = $\frac{E}{2(1+\nu)}$

β : shape factor of section

If we specialize to

$$u \equiv 0, \quad k \equiv 0, \quad N \equiv 0$$

then we get:

$$\frac{\partial}{\partial x} (\beta A_0 G (\frac{\partial W}{\partial x} - \psi)) + p = \ddot{W} m_0 \quad (21.92)$$

$$\frac{\partial}{\partial x} (EI_0 \frac{\partial \psi}{\partial x}) + \beta A_0 G (\frac{\partial W}{\partial x} - \psi) + h = -m_0 r^2 \ddot{\psi}$$

two equations for the unknowns

$$W = W(x, t), \quad \psi = \psi(x, t)$$

If the beam is uniform, β, A_0, I_0, r all independent of x , then we may eliminate ψ and get the single equation in W :

$$\frac{\partial^4 W}{\partial x^4} - \left(\frac{m_0}{\beta A_0 G} + \frac{m_0 r^2}{EI_0} \right) \frac{\partial^4 W}{\partial x^2 \partial t^2} + \frac{m_0}{EI_0} \frac{\partial^2 W}{\partial t^2} + \quad (21.93)$$

$$\frac{m_0}{\beta A_0 G} \frac{m_0 r^2}{EI_0} \frac{\partial^4 W}{\partial t^4} = \frac{1}{EI_0} \left(p + \frac{\partial h}{\partial x} \right)$$

$$\frac{m_0}{\beta A_0 G} + \frac{m_0 r^2}{EI_0} = \frac{m_0}{EA_0} \left(\frac{2(1+\nu)}{\beta} + 1 \right) =$$

$$\frac{\rho_0}{E} \left(\frac{2(1+\nu)}{\beta} + 1 \right), \quad m_0 = \rho_0 A_0$$

$$\frac{m_0}{\beta A_0 G} \frac{m_0 r^2}{EI_0} = \left(\frac{m_0}{A_0} \right)^2 \frac{1}{E^2} \frac{2(1+\nu)}{\beta} =$$

$$\left(\frac{\rho_0}{E} \right)^2 \frac{2(1+\nu)}{\beta}$$

where ρ_0 is the mass density.

$$\frac{m_0}{EI_0} = \frac{m_0}{EA_0 r^2} = \frac{\rho_0}{E} \frac{1}{r^2}$$

For free vibrations with harmonic motion

$$W(x, t) = \hat{W}(x) \sin \omega t \quad (21.94)$$

We get, from (21.93), the following equation of motion

$$\frac{d^4 \hat{W}}{dx^4} + \omega^2 \left\{ \left(\frac{m_0}{\beta A_0 G} + \frac{m_0 r^2}{EI_0} \right) \frac{d^2 \hat{W}}{dx^2} - \frac{m_0}{EI_0} \hat{W} \right\} +$$

$$\omega^4 \frac{m_0^2 r^2}{\beta A_0 G EI_0} \hat{W} = 0 \quad (21.95)$$

We do not take this matter any further here. The interested reader may look up pages 208-212 in Geradin & Rixen for some specific applications.

In general we have the conclusion that the shear deflection and the rotatory inertia will both decrease the eigentrequencies, the shear deflection to a larger extent. These effects are more pronounced for beams with hollow (thin-walled) sections than for those with solid sections.

LECTURE 16

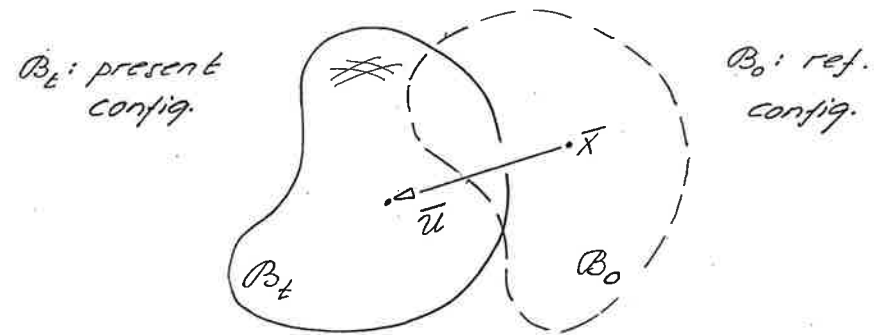
CONTINUOUS SYSTEMS

The three-dimensional body
 Navier's equation of motion
 Longitudinal and transverse waves
 The acoustic tensor

22. Three-dimensional bodies

In this chapter we consider, very briefly, fully three-dimensional bodies and their vibrational properties.

We use some basic equations from continuum mechanics touched upon in Lecture 1, Ex 3.



The motion of the body is described by the displacement:

$$\vec{u} = \vec{u}(\vec{X}, t), \quad \vec{X} \in B_0 \quad (22.1)$$

from the "reference configuration" B_0 to the "present configuration" B_t .

The equations of motion are given by:

$$\text{div } \underline{I} + \bar{b} = \rho \ddot{\bar{u}} \quad (22.2)$$

$\underline{I}$ is (the symmetric) stress tensor.

$\bar{b}$ is the external body force.

ρ is the mass density.

For a "linear elastic and isotropic material" we have the constitutive relation:

$$\underline{I} = \lambda (\text{tr } \underline{E}) \underline{1} + 2\mu \underline{E} \quad (22.3)$$

where $\underline{E}$, "the strain tensor" is defined by:

$$\underline{E} = \frac{1}{2} (\nabla \bar{u} + \nabla \bar{u}^T), \quad \underline{E}^T = \underline{E} \quad (22.4)$$

of $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ ON-base)

$$\bar{u} = \bar{e}_1 u_1 + \bar{e}_2 u_2 + \bar{e}_3 u_3$$

$$u_i = u_i(\bar{x}, t), \quad u_2 = u_2(\bar{x}, t), \quad u_3 = u_3(\bar{x}, t)$$

$$\text{tr}(\underline{E}) = \text{"trace of } \underline{E}\text{"} = E_{11} + E_{22} + E_{33}$$

$$\bar{x} = \bar{e}_1 x_1 + \bar{e}_2 x_2 + \bar{e}_3 x_3$$

then the components of $\underline{E}$ are:

$$E_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (22.5)$$

The material constants λ and μ are called "Lame' moduli". They are related to the modulus of elasticity E and Poisson's number ν :

$$\lambda = E \frac{\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \quad (22.6)$$

μ is also called the shear modulus and is sometimes denoted G .

By substituting (22.3) into (22.2)

we get Navier's equation of motion:

$$\mu \Delta \bar{u} + (\lambda + \mu) \nabla \text{div} \bar{u} + \bar{b} = \rho \ddot{\bar{u}} \quad (22.7)$$

where Δ is the "laplace operator":

Note that:

$$\Delta \underline{u} = \text{div} \nabla \underline{u}, \quad \text{div} (\nabla \nabla^T) = \nabla \text{div} \underline{u}$$

$$\text{curl} \text{curl} \underline{u} = \nabla \text{div} \underline{u} - \Delta \underline{u}$$

From the last identity we conclude

that (22.7) is equivalent to:

$$(22.8) \quad (\lambda + 2\mu) \nabla \text{div} \underline{u} - \mu \text{curl} \text{curl} \underline{u} + \bar{b} = \rho \ddot{\underline{u}}$$

We introduce the constants:

$$(22.9) \quad c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}, \quad -1 < \nu < 0.5$$

$$c_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{E}{2(1+\nu)\rho}}, \quad E > 0$$

Then (22.8) becomes:

$$(22.10) \quad c_L^2 \nabla \text{div} \underline{u} - c_T^2 \text{curl} \text{curl} \underline{u} + \bar{b} = \ddot{\underline{u}}$$

Introducing the wave operators:

$$(22.11) \quad \square_L \underline{u} = c_L^2 \Delta \underline{u} - \ddot{\underline{u}}, \quad \square_T \underline{u} = c_T^2 \Delta \underline{u} - \ddot{\underline{u}}$$

Navier's equation (22.10) =

$$(22.12) \quad \square_L \underline{u} + (c_L^2 - c_T^2) \text{curl} \text{curl} \underline{u} + \bar{b} = \ddot{\underline{u}}$$

Navier's equation (22.8) =

$$(22.13) \quad \square_T \underline{u} + (c_L^2 - c_T^2) \nabla \text{div} \underline{u} + \bar{b} = \ddot{\underline{u}}$$

From these we may conclude that

for irrotational motion: $\text{curl} \underline{u} = \bar{0}$

we have: $(\bar{b} = \bar{0})$

$$(22.14) \quad \square_L \underline{u} = \bar{0} \Leftrightarrow \Delta \underline{u} - \frac{1}{c_L^2} \frac{\partial^2 \underline{u}}{\partial t^2} = \bar{0}$$

i.e. a wave-equation for the dis-

placement $\underline{u}$. The corresponding wave-

speed c_L is called longitudinal or

irrotational.

For isochoric motion: $\text{div } \bar{u} = 0$

We have:

$$\| \square_T \bar{u} = \bar{0} \Leftrightarrow \Delta \bar{u} - \frac{1}{C_T^2} \frac{\partial^2 \bar{u}}{\partial t^2} = \bar{0} \quad (22.15)$$

The wave-speed C_T is called transverse or isochoric.

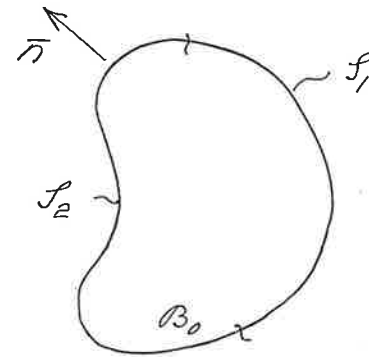
using the fact that $\text{div curl} = 0$,
 $\text{curl grad} = \bar{0}$ we find, from Navier's equation that,

$$\| \begin{aligned} \square_L \text{div } \bar{u} &= 0 \\ \square_T \text{curl } \bar{u} &= \bar{0} \end{aligned} \quad (22.16)$$

i.e. the vector-field $\text{div } \bar{u}$ satisfies a wave-eq. with wave-speed C_L , while $\text{curl } \bar{u}$ satisfies a wave-eq. with speed C_T . Note that

$$\left| \frac{C_L}{C_T} = \sqrt{\frac{2(1-\nu)}{1-2\nu}} \quad \nu = \frac{1}{3} \Rightarrow C_L = 2C_T \right.$$

Boundary and initial values:



Boundary:

$$\partial B = S_1 \cup S_2$$

$$S_1 \cap S_2 = \emptyset$$

Displacement condition:

$$\bar{u} = \hat{\bar{u}}, \quad x \in S_1 \quad (22.17)$$

where $\hat{\bar{u}} = \hat{\bar{u}}(\bar{x}, t)$, $\bar{x} \in S_1$ is a prescribed displacement on the part S_1 of the boundary of the body.

Traction condition:

$$I \bar{n} = \hat{\bar{t}}, \quad \bar{x} \in S_2 \quad (22.18)$$

where $\hat{\bar{t}} = \hat{\bar{t}}(\bar{x}, t)$, $\bar{x} \in S_2$ is a prescribed traction on the part S_2 of the boundary of the body.

Some math. notation:

$$(\nabla \bar{u})_{ij} = \frac{\partial u_j}{\partial x_i}$$

$$\text{div } \bar{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{tr}(\nabla \bar{u})$$

$$(\text{curl } \bar{u})_{ij} = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

$$\epsilon_{ijk} = \begin{cases} -1 & (ijk) \text{ odd perm.} \\ 1 & (ijk) \text{ even perm.} \\ 0 & \text{otherwise} \end{cases}$$

$$(\text{div } \underline{I})_i = \frac{\partial T_{i1}}{\partial x_1} + \frac{\partial T_{i2}}{\partial x_2} + \frac{\partial T_{i3}}{\partial x_3}$$

$$\Delta \bar{u} = \bar{e}_1 \Delta u_1 + \bar{e}_2 \Delta u_2 + \bar{e}_3 \Delta u_3$$

$$\Delta u_i = \frac{\partial^2 u_i}{\partial x_1^2} + \frac{\partial^2 u_i}{\partial x_2^2} + \frac{\partial^2 u_i}{\partial x_3^2}$$

$$\text{tr } \underline{I} = T_{11} + T_{22} + T_{33} \quad \text{the trace}$$

$$(\bar{a} \otimes \bar{b}) \bar{u} = \bar{a} \bar{b} \cdot \bar{u} \quad \text{the tensor product}$$

Wave propagation

A progressive plane wave is given by

$$\bar{u}(\bar{x}, t) = \bar{a} F(\bar{m} \cdot \bar{x} - ct), \quad \bar{a} \neq \bar{0} \quad (22.19)$$

where $F = F(s)$ is a given function (wave form). $\bar{a}$ and $\bar{m}$ are unit vectors and c is the wave propagation speed.

From (22.19) it follows that:

$$\begin{aligned} \nabla \bar{u} &= \bar{a} \otimes \frac{dF}{ds} \nabla(\bar{m} \cdot \bar{x} - ct) = F' \bar{a} \otimes \bar{m} \\ \text{div } \bar{u} &= F' \bar{a} \cdot \bar{m} = \text{tr}(\nabla \bar{u}) \\ \text{curl } \bar{u} &= F' \bar{m} \times \bar{a} \end{aligned} \quad (22.20)$$

We say that the wave is longitudinal if:

$$\bar{a} \parallel \bar{m} \quad (\bar{a} = \bar{m} \text{ or } \bar{a} = -\bar{m})$$

and transverse if:

$$\bar{a} \perp \bar{m} \quad (\bar{a} \cdot \bar{m} = 0)$$

Note that the material velocity and

acceleration are given by:

$$\|\ddot{\vec{u}} = \vec{a}(1 - cF'), \quad \ddot{\vec{u}} = \vec{a}c^2F'' \quad (22.21)$$

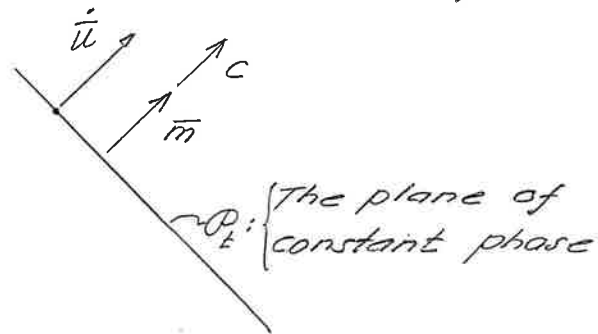
We here assume that:

$$F'' \neq 0$$

The propagation of the wave is given by the moving plane of constant phase:

$$P_t = \{ \vec{x} : \vec{m} \cdot \vec{x} - ct = \text{const.} \}$$

which represents a moving plane in space with normal velocity $c\vec{m}$.



For a longitudinal wave we have

$$\|\vec{a} \parallel \vec{m}, \quad \vec{u} \parallel \vec{m}$$

and for a transverse wave:

$$\|\vec{a} \perp \vec{m}, \quad \vec{u} \perp \vec{m}$$

It also follows that:

Longitudinal wave $\Leftrightarrow$

$$\text{curl } \vec{u} = F' \vec{m} \times \vec{a} = \vec{0}$$

Transverse wave $\Leftrightarrow \text{div } \vec{u} = F' \vec{a} \cdot \vec{m} = 0$

Navier's equation ($\vec{b} = \vec{0}$):

$$\Delta_t \vec{u} + (c_L^2 - c_T^2) \nabla(\text{div } \vec{u}) = \vec{0}$$

But

$$\Delta \vec{u} = \text{div}(\nabla \vec{u}) = \text{div}(F' \vec{a} \otimes \vec{m}) =$$

$$\vec{a} \otimes \vec{m} \nabla F' = (\vec{0} \otimes \vec{m}) \vec{m} F'' = \vec{a} F''$$

$$\nabla(\text{div } \vec{u}) = \nabla(F' \vec{a} \cdot \vec{m}) = \vec{m} F'' \vec{a} \cdot \vec{m}$$

This implies:

$$\vec{a} c_T^2 F'' - \vec{a} c^2 F'' + (c_L^2 - c_T^2) \vec{m} \vec{a} \cdot \vec{m} F'' = 0$$

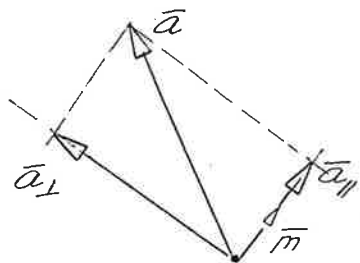
$$F'' \neq 0 \Rightarrow$$

$$\|\vec{a} c_T^2 - \vec{a} c^2 + (c_L^2 - c_T^2) \vec{m} \vec{a} \cdot \vec{m} = 0 \quad (22.22)$$

This equation has to be satisfied

by $\bar{a}$, $\bar{m}$ and c in order to make the plane wave (22.19) possible. We may write

$$\bar{a} = \bar{a}_{\parallel} + \bar{a}_{\perp}, \quad \bar{a}_{\parallel} = \bar{m} \bar{a} \cdot \bar{m}$$



$$\bar{a}_{\perp} \cdot \bar{a}_{\parallel} = 0$$

Then from (22.22) it follows that

$$(c_L^2 - c^2) \bar{a}_{\parallel} + (c_T^2 - c^2) \bar{a}_{\perp} = \bar{0}$$

If $\bar{a}_{\parallel} \neq \bar{0}$ then $(c_L^2 - c^2) \bar{a}_{\parallel} \cdot \bar{a}_{\parallel} = 0$ and from this we conclude that

$$c = c_L \text{ and } \bar{a}_{\perp} = \bar{0}$$

i.e. $\bar{a} = \bar{a}_{\parallel} \parallel \bar{m}$ and we have a longitudinal wave.

If $\bar{a}_{\parallel} = \bar{0}$ then, since $\bar{a}_{\perp} \neq \bar{0}$,

it follows that

$$c = c_T$$

i.e. $\bar{a} \cdot \bar{m} = 0$ and we have a transverse wave.

Thus for an isotropic elastic material only two types of plane waves are possible; longitudinal and transverse.

If the material is an-isotropic we may write the constitutive eq.

$$\underline{I} = \mathbb{C}[\underline{E}] \quad (22.23)$$

where $\mathbb{C}$ is a fourth order tensor with components: C_{ijkl} ($T_{ij} = C_{ijkl} E_{kl}$, $C_{ijkl} = C_{jikl} = C_{jilk}$). The strain tensor:

$$\underline{E} = \frac{1}{2} F'(\bar{a} \otimes \bar{m} + \bar{m} \otimes \bar{a}) = \nabla$$

$$\mathbb{C}[\underline{E}] = F' \mathbb{C}[\bar{a} \otimes \bar{m}]$$

$$\text{div } \underline{I} = \text{div } \mathbb{C}[\underline{E}] = \text{div}(F' \mathbb{C}[\bar{a} \otimes \bar{m}]) =$$

$$F'' \mathcal{C}[\bar{a} \otimes \bar{m}] \bar{m}.$$

Eg. of motion: ($\bar{b} = \bar{0}$)

$$\text{div } I = \rho \ddot{u} \quad \Rightarrow$$

$$F'' \mathcal{C}[\bar{a} \otimes \bar{m}] \bar{m} = c^2 F'' \bar{a}$$

$$F'' \neq 0 \quad \Rightarrow$$

$$\mathcal{C}[\bar{a} \otimes \bar{m}] \bar{m} = \rho c^2 \bar{a} \quad (22.24)$$

The tensor

$$\underline{A} = \underline{A}(\bar{m}) \quad (22.25)$$

defined by

$$\underline{A} \bar{a} = \frac{1}{\rho} \mathcal{C}[\bar{a} \otimes \bar{m}] \bar{m} \quad (22.26)$$

is called the acoustic tensor.

The equation of motion for a progressive plane wave implies:

$$\underline{A}(\bar{m}) \bar{a} = c^2 \bar{a} \quad (22.27)$$

Thus for a plane wave to propagate in the direction $\bar{m}$ with amplitude (direction) $\bar{a}$, $\bar{a}$ has to be an eigenvector to the acoustic tensor $\underline{A}(\bar{m})$ and the square of the wave propagation speed c^2 must be the corresponding eigenvalue.

If $\mathcal{C}$ is symmetric and positive definite it may be shown that

$\underline{A}(\bar{m})$ is symmetric and pos. definite.

We thus have three orthogonal eigenvectors $\bar{a}_1, \bar{a}_2, \bar{a}_3$ and three associated positive eigenvalues c_1^2, c_2^2, c_3^2 for every propagation direction $\bar{m}$.

$$\underline{A}(\bar{m}) \bar{a}_i = c_i^2 \bar{a}_i \quad i = 1, 2, 3 \quad (22.28)$$

Note that for an isotropic material:

$$A(\bar{m}) = C_L^2 \bar{m} \otimes \bar{m} + C_T^2 (1 - \bar{m} \otimes \bar{m})$$

and the acoustic tensor has the eigenvectors:

$$\bar{m} \text{ with eigenvalue } C_L^2$$

$$\bar{n}, \bar{n} \cdot \bar{m} = 0 \text{ with eigenvalue } C_T^2$$

"The acoustic tensor" is a tensor (linear mapping) since:

$$\frac{1}{\rho} \mathbb{C}[(\bar{a} + \bar{b}) \otimes \bar{m}] \bar{m} = \frac{1}{\rho} \mathbb{C}[\bar{a} \otimes \bar{m} +$$

$$\bar{b} \otimes \bar{m}] \bar{m} = \frac{1}{\rho} \mathbb{C}[\bar{a} \otimes \bar{m}] \bar{m} +$$

$$\frac{1}{\rho} \mathbb{C}[\bar{b} \otimes \bar{m}] \bar{m}$$

$$\frac{1}{\rho} \mathbb{C}[\lambda \bar{a} \otimes \bar{m}] \bar{m} = \lambda \frac{1}{\rho} \mathbb{C}[\bar{a} \otimes \bar{m}] \bar{m}$$

where we have employed the linearity of $\mathbb{C}$.

Free vibrations

Assume a motion of the form:

$$\bar{u}(\bar{x}, t) = \hat{\bar{u}}(\bar{x}) \sin \omega t \quad (22.29)$$

Then

$$\underline{\underline{E}}(\bar{x}, t) = \underline{\underline{E}}^{\hat{}}(\bar{x}) \sin \omega t$$

$$\underline{\underline{E}}^{\hat{}} = \frac{1}{2} (\nabla \hat{\bar{u}} + \nabla \hat{\bar{u}}^T)$$

$$\underline{\underline{I}}(\bar{x}, t) = \underline{\underline{I}}^{\hat{}}(\bar{x}) \sin \omega t$$

and the amplitudes (modes) $\hat{\bar{u}}$, $\underline{\underline{E}}^{\hat{}}$ and $\underline{\underline{I}}^{\hat{}}$ satisfy:

$$\underline{\underline{I}}^{\hat{}} = \mathbb{C}[\underline{\underline{E}}^{\hat{}}] \quad (22.30)$$

$$\operatorname{div} \underline{\underline{I}}^{\hat{}} + \rho \omega^2 \hat{\bar{u}} = \bar{0}$$

For an isotropic material ($\mathbb{C}[\underline{\underline{E}}] = \lambda \operatorname{tr} \underline{\underline{E}} + 2\mu \underline{\underline{E}}$) we have:

$$\mu \Delta \hat{\bar{u}} + (\lambda + \mu) \nabla \operatorname{div} \hat{\bar{u}} + \rho \omega^2 \hat{\bar{u}} = \bar{0} \quad (22.31)$$

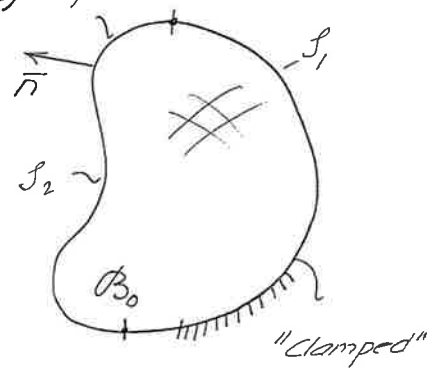
the so-called Navier's eq.

Boundary conditions ("clamped-free")

$$\hat{u} = \bar{0} \quad \text{on } \mathcal{I}_1 \quad (22.32) \text{ "free"}$$

$$\hat{E} = I\bar{n} = \bar{0} \quad \text{on } \mathcal{I}_2$$

One part of the boundary $\mathcal{I}_1$ is clamped ("essential condition") and the other part $\mathcal{I}_2$ is free ("natural boundary condition")



We introduce the short hand notation:

$$\mathcal{L}\hat{u} = \frac{1}{\rho} \operatorname{div} [\mathcal{E}] = \frac{1}{\rho} \operatorname{div} [\nabla \hat{u}] \quad (22.33)$$

$\mathcal{L}$ is a linear differential operator.

We introduce the "scalar products":

$$\langle \hat{u}, \hat{v} \rangle = \int_{B_0} \hat{u}(\bar{x}) \cdot \hat{v}(\bar{x}) \rho(\bar{x}) dV(\bar{x})$$

$$\langle \hat{u}, \hat{v} \rangle_c = \int_{B_0} \nabla \hat{u} \cdot [\nabla \hat{v}] \rho dV$$

Then

$$\langle \mathcal{L}\hat{u}, \hat{v} \rangle = \int_{B_0} (\operatorname{div} [\mathcal{E}[\nabla \hat{u}]] \cdot \hat{v}) dV =$$

$$\int_{B_0} (\operatorname{div} (\mathcal{E}[\nabla \hat{u}] \hat{v}) - \nabla \hat{v} \cdot \mathcal{E}[\nabla \hat{u}]) dV =$$

$$- \langle \hat{u}, \hat{v} \rangle_c + \int_{\partial B_0} \underbrace{(\mathcal{E}[\nabla \hat{u}] \bar{n})}_{\hat{E} = \hat{E}(\hat{u})} \cdot \hat{v} da =$$

$$- \langle \hat{u}, \hat{v} \rangle_c + \int_{\partial B_0} \hat{E} \cdot \hat{v} da.$$

Now if $\hat{u}, \hat{v}$ satisfy the boundary conditions (22.32), then:

$$\langle \mathcal{L}\hat{u}, \hat{v} \rangle = - \langle \hat{u}, \hat{v} \rangle_c$$

and since $\mathcal{E}$ is symmetric:

$$\langle \mathcal{L}\hat{u}, \hat{v} \rangle = \langle \hat{u}, \mathcal{L}\hat{v} \rangle \quad (22.34)$$

$\mathcal{L}$ is a symmetric operator (compare with the symmetric stiffness matrix $\underline{K}$)

The eigenvalue problem for the "free vibrations" is formulated

according to:

Find $\lambda \in \mathbb{R}$, $\hat{u} = \hat{u}(x) \neq \bar{0}$

such that

$$\mathcal{L}\hat{u} + \lambda\hat{u} = \bar{0} \quad (22.35)$$

$$\hat{u} = \bar{0} \quad \text{on } \mathcal{I}_1$$

$$\hat{u} = \bar{0} \quad \text{on } \mathcal{I}_2$$

It follows directly from (22.35) and the fact that $\mathcal{G}$ is pos. definite:

$$0 \leq \langle \hat{u}, \hat{u} \rangle_{\mathcal{G}} = - \langle \mathcal{L}\hat{u}, \hat{u} \rangle = \lambda \langle \hat{u}, \hat{u} \rangle$$

$$\lambda = \frac{\langle \hat{u}, \hat{u} \rangle_{\mathcal{G}}}{\langle \hat{u}, \hat{u} \rangle} \geq 0, \quad (\lambda = \omega^2) \quad (22.36)$$

i.e. the eigenvalues are positive!

Furthermore let $\lambda_1, \hat{u}_1$ and $\lambda_2, \hat{u}_2$ be eigensolutions to (22.35). Then

$$\mathcal{L}\hat{u}_1 + \lambda_1\hat{u}_1 = \bar{0}, \quad \mathcal{L}\hat{u}_2 + \lambda_2\hat{u}_2 = \bar{0}$$

and

$$\langle \mathcal{L}\hat{u}_1, \hat{u}_2 \rangle + \lambda_1 \langle \hat{u}_1, \hat{u}_2 \rangle = 0$$

$$\langle \mathcal{L}\hat{u}_2, \hat{u}_1 \rangle + \lambda_2 \langle \hat{u}_2, \hat{u}_1 \rangle = 0$$

But $\mathcal{L}$ is self-adjoint $\Rightarrow$

$$\langle \mathcal{L}\hat{u}_1, \hat{u}_2 \rangle = \langle \mathcal{L}\hat{u}_2, \hat{u}_1 \rangle \Rightarrow$$

$$(\lambda_1 - \lambda_2) \langle \hat{u}_1, \hat{u}_2 \rangle = 0$$

and consequently:

$$\lambda_1 \neq \lambda_2 \Rightarrow \langle \hat{u}_1, \hat{u}_2 \rangle = \langle \hat{u}_1, \hat{u}_2 \rangle_{\mathcal{G}} = 0 \quad (22.37)$$

i.e. modes belonging to different eigenvalues are orthogonal!

Let:

$$\bar{u}_1, \bar{u}_2, \dots$$

$$\lambda_1, \lambda_2, \dots \quad (0 < \lambda_1 \leq \lambda_2 \leq \dots)$$

be a system of (normal) modes then it may be shown (under specified conditions) that

$$\lim_{n \rightarrow \infty} \lambda_n = \infty$$

and if $\bar{F} = \bar{F}(\bar{x})$ is a function on B which fulfills the essential boundary condition: $\bar{F}(\bar{x}) = \bar{0}$ on S , then

$$\lim_{n \rightarrow \infty} \left\| \bar{F} - \sum_{i=1}^n c_i \hat{u}_i \right\| = 0$$

where (the Fourier-coefficients)

$$c_i = \langle \bar{F}, \hat{u}_i \rangle$$

Here the norm $\|\bar{g}\|$ is defined as

$$\|\bar{g}\|^2 = \langle \bar{g}, \bar{g} \rangle$$

Now if we look at the extended dynamic problem ($\bar{0} \neq \bar{0}$):

$$\mathcal{L}\bar{u} + \frac{\bar{0}}{\rho} = \ddot{\bar{u}} \quad (22.38)$$

with boundary conditions according

to (22.32) we may look for a solution on the form (mode decomposition):

$$\bar{u}(x, t) = \sum_{i=1}^{\infty} \hat{u}_i(x) \eta_i(t) \quad (22.39)$$

where the functions $\eta_i = \eta_i(t)$ may be comprehended as normal coordinates. Insertion of (22.39) into (22.38) gives:

$$\sum_{i=1}^{\infty} \mathcal{L} \hat{u}_i \eta_i + \frac{\bar{0}}{\rho} = \sum_{i=1}^{\infty} \hat{u}_i \ddot{\eta}_i$$

Multiply this with $\rho \hat{u}_k$ and integrate over the body:

$$\sum_{i=1}^{\infty} \int_{B_0} \hat{u}_k \cdot \mathcal{L} \hat{u}_i \eta_i \rho \, dV + \int_{B_0} \hat{u}_k \cdot \bar{0} \, dV =$$

$$\sum_{i=1}^{\infty} \int_{B_0} \hat{u}_k \cdot \hat{u}_i \ddot{\eta}_i \rho \, dV$$

But

$$\mathcal{L}\hat{u}_i = -\lambda_i \hat{u}_i = -\omega_i^2 \hat{u}_i$$

and using the orthogonality we get:

$$-(\int_{\Omega_0} \hat{u}_k \cdot \hat{u}_k \rho dv) \lambda_k \eta_k + \int_{\Omega_0} \hat{u}_k \cdot \bar{b} dv =$$

$$(\int_{\Omega_0} \hat{u}_k \cdot \hat{u}_k \rho dv) = 0$$

$$\ddot{\eta}_k + \omega_k^2 \eta_k = \frac{\int_{\Omega_0} \hat{u}_k \cdot \bar{b} dv}{\int_{\Omega_0} \hat{u}_k \cdot \hat{u}_k \rho dv} = \frac{\int_{\Omega_0} \hat{u}_k \cdot \bar{b} dv}{\|\hat{u}_k\|^2} \quad (22.39)$$

put

$$\phi_k(t) = \frac{\int_{\Omega_0} \hat{u}_k(x) \cdot \bar{b}(x, t) dv}{\|\hat{u}_k\|^2}$$

then

$$\ddot{\eta}_k + \omega_k^2 \eta_k = \phi_k, \quad k=1, 2, \dots \quad (22.40)$$

Initial data:

$$\bar{u}(x, 0) = \bar{u}_0(x)$$

$$\dot{\bar{u}}(x, 0) = \dot{\bar{u}}_0(x)$$

$$\bar{u}_0(x) = \sum_{i=1}^{\infty} \hat{u}_i(x) \eta_i(0) = 0$$

$$\eta_i(0) = \langle \hat{u}_i, \bar{u}_0 \rangle = \int_{\Omega_0} \hat{u}_i \cdot \bar{u}_0 \rho dv \quad (22.41)$$

$$\dot{\bar{u}}_0(x) = \sum_{i=1}^{\infty} \hat{u}_i \dot{\eta}_i(0) = 0$$

$$\dot{\eta}_i(0) = \langle \hat{u}_i, \dot{\bar{u}}_0 \rangle = \int_{\Omega_0} \hat{u}_i \cdot \dot{\bar{u}}_0 \rho dv$$

The solution of (22.40) :

$$\eta_i(t) = \eta_i(0) \cos \omega_i t + \frac{\dot{\eta}_i(0)}{\omega_i} \sin \omega_i t +$$

$$\frac{1}{\omega_i} \int_0^t \phi_i(\tau) \sin \omega_i(t-\tau) d\tau \quad (22.42)$$

The solution may now be written:

$$\bar{u}(\bar{x}, t) = \sum_{i=1}^{\infty} \hat{u}_i(x) \int_{\Omega_0} \hat{u}_i \cdot \bar{u}_0 \rho dv \cos \omega_i t +$$

$$\sum_{i=1}^{\infty} \hat{u}_i(x) \int_{\Omega_0} \hat{u}_i \cdot \dot{\bar{u}}_0 \rho dv \frac{1}{\omega_i} \sin \omega_i t +$$

$$\sum_{i=1}^{\infty} \hat{u}_i(x) \frac{1}{\omega_i} \int_0^t \hat{f}_i(z) \sin \omega_i (t-z) dz$$

(22.43)



Claude Louis Marie Henri Navier

1785 - 1836

LECTURE 17

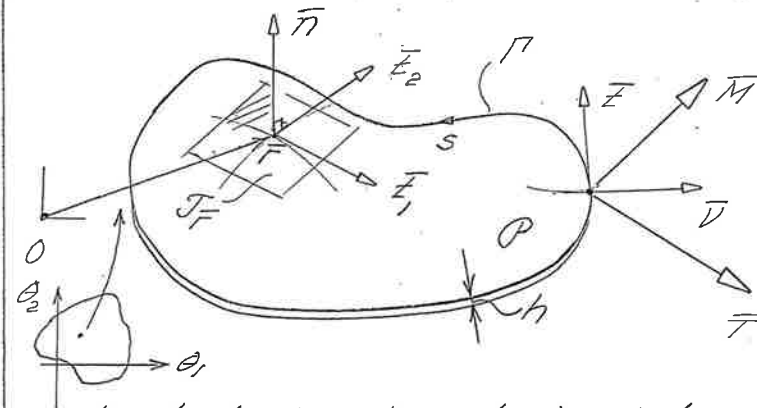
CONTINUOUS SYSTEMS

The two-dimensional body
Differential geometry

23. Two-dimensional bodies

In this chapter we look at some simple cases of two-dimensional bodies; the membrane and the thin plate.

Consider a two-dimensional body $\mathcal{B}$. We make a "free-body" diagram of a part of the body $\mathcal{P}$ according to the figure below:



The body is characterized by its "mean surface" which is given as a space-surface $\mathcal{S}$:

$$\vec{F} = \vec{F}(\theta_1, \theta_2, t), \quad (\theta_1, \theta_2) \in \mathbb{R}^2 \quad (23.1)$$

where

θ_1, θ_2 are local coordinates of the surface; and a thickness parameter h .

The vectors:

$$\bar{E}_1 = \frac{\partial \bar{r}}{\partial \theta_1}, \quad \bar{E}_2 = \frac{\partial \bar{r}}{\partial \theta_2}, \quad \bar{E}_1 \times \bar{E}_2 \neq \bar{0} \quad (23.2)$$

are tangent vectors to S . They belong to the tangent space $\mathcal{T}_{\bar{r}}$ at the position $\bar{r}$ on the surface:

$$\bar{E}_1, \bar{E}_2 \in \mathcal{T}_{\bar{r}} = \mathcal{T}_{\bar{r}}(t) \quad (23.3)$$

The unit normal to the surface is given by:

$$\bar{n} = \frac{\bar{E}_1 \times \bar{E}_2}{|\bar{E}_1 \times \bar{E}_2|}, \quad \bar{n} \cdot \bar{E}_1 = \bar{n} \cdot \bar{E}_2 = 0 \quad (23.4)$$

Let Γ denote the boundary curve separating the body part $\mathcal{D}$ from the rest of the two-dimensional body $\mathcal{B}$. Let

$$\bar{v} = \bar{v}(\bar{r}, t), \quad \bar{r} \in \Gamma, \quad \bar{v}(\bar{r}, t) \in \mathcal{T}_{\bar{r}}(t) \quad (23.5)$$

denote the outward unit normal vector field to Γ which is tangential to S :

$$\bar{v}(\bar{r}, t) \in \mathcal{T}_{\bar{r}}(t) \quad \Leftrightarrow \quad \bar{n} \cdot \bar{v} = 0 \quad (23.6)$$

The internal forces acting from $\mathcal{B} \setminus \mathcal{D}$ on $\mathcal{D}$ (across Γ) are the following two vector fields measured per unit length of the boundary curve Γ :

The traction: $[\text{Nm}^{-1}]$

$$\bar{T} = \bar{T}(\bar{r}, t), \quad \bar{r} \in \Gamma \quad (23.7)$$

The moment: $[\text{Nmm}^{-1}]$

$$\bar{M} = \bar{M}(\bar{r}, t), \quad \bar{r} \in \Gamma \quad (23.8)$$

The external forces are given by:

The external force density: $[\text{Nm}^{-2}]$

$$\bar{K} = \bar{K}(\bar{r}, t), \quad \bar{r} \in S \quad (23.9)$$

The external moment density: $[\text{Nmm}^{-2}]$

$$\bar{H} = \bar{H}(\bar{r}, t), \quad \bar{r} \in S \quad (23.10)$$

These two vector fields are given as densities, measured per unit area of the mean-surface S .

We now formulate:

The equations of motion of the two-

dimensional body $\mathcal{B}$:

$$(23.11) \quad \int_S \bar{K} da + \int_S \bar{T} ds = \int_S \bar{a} m da$$

$$\int_S (\bar{F} \times \bar{K} + \bar{H}) da + \int_S (\bar{F} \times \bar{T} + \bar{M}) ds =$$

$$(23.12) \quad \int_S \bar{F} \times \bar{a} m da$$

where da = surface area element.

$\bar{a} = \ddot{\bar{r}}$: material (at mean-surface) acceleration.

$m = m(\bar{r}, t)$: mass density (mass per

unit area of S).

We now assume that there is a tensor-

field $\bar{I}$ on S such that:

$$(23.13) \quad \bar{T} = \bar{I} \bar{v}$$

and a tensor field $\bar{M}$ such that

$$(23.14) \quad \bar{M} = \bar{M} \bar{v}$$

i.e. both $\bar{T}$ and $\bar{M}$ are assumed to depend linearly on the outward unit

normal vector $\bar{v}$. $\bar{I}$ is called the

traction tensor and $\bar{M}$ the moment

tensor. With these tensors the eq. of motion may be written:

$$(23.15) \quad \int_S \bar{K} da + \int_S \bar{I} \bar{v} ds = \int_S \bar{a} m da$$

$$\int_S (\bar{F} \times \bar{K} + \bar{H}) da + \int_S (\bar{F} \times \bar{I} \bar{v} + \bar{M} \bar{v}) ds$$

$$(23.16) \quad \int_S \bar{F} \times \bar{a} m da$$

Note that we have not included rota-

tional inertia of the surface in the

moment of momentum balance. This

may be appropriate for thin bodies.

Compare the equations (23.15-16) with

the corresponding ones for 1-dim. bodies!

By using the "surface-divergence"
theorem we have:

$$\int_{\Pi} \underline{I} \underline{V} ds = \int_S \text{div}_S \underline{I} da \quad (23.17)$$

$$\int_{\Pi} (\underline{F} \times \underline{I} \underline{V} + \underline{M} \underline{V}) ds = \int_S (\text{div}_S (\underline{F} \times \underline{I}) + \text{div}_S \underline{M}) da \quad (23.18)$$

This inserted into (23.15-16) gives:

$$\int_S (\underline{K} + \text{div}_S \underline{I} - \bar{a} m) da = \bar{O} \quad (23.19)$$

$$\int_S (\underline{F} \times \underline{K} + \bar{H} + \text{div}_S (\underline{F} \times \underline{I}) + \text{div}_S \underline{M} - \underline{F} \times \bar{a} m) da = \bar{O} \quad (23.20)$$

and the local equations:

$$\underline{K} + \text{div}_S \underline{I} = \bar{a} m \quad (23.21)$$

$$\underline{F} \times \underline{K} + \bar{H} + \text{div}_S (\underline{F} \times \underline{I}) + \text{div}_S \underline{M} = \underline{F} \times \bar{a} m \quad (23.22)$$

Now it may be shown that (see forthcoming remark on diff. geometry):

$$\text{div}_S (\underline{F} \times \underline{I}) = \underline{F} \times \text{div}_S \underline{I} - 2 \text{ax}(\underline{V}_S \underline{F} \underline{I}^T) \quad (23.23)$$

(where $\text{ax}(\underline{A})$ denotes the axial vector corresponding to $\underline{A}$.)

Then (23.22) may be written:

$$\underline{F} \times \underline{K} + \bar{H} + \underline{F} \times \text{div}_S \underline{I} - 2 \text{ax}(\underline{V}_S \underline{F} \underline{I}^T) + \text{div}_S \underline{M} = \underline{F} \times \bar{a} m \quad (23.24)$$

or

$$\underline{F} \times (\underline{K} + \text{div}_S \underline{I} - \bar{a} m) + \bar{H} - 2 \text{ax}(\underline{V}_S \underline{F} \underline{I}^T) + \text{div}_S \underline{M} = \bar{O} \quad (23.25)$$

using

$$\underline{K} + \text{div}_S \underline{I} = \bar{a} m \quad (23.26)$$

this reduces to:

$$\bar{H} - 2 \text{ax}(\underline{V}_S \underline{F} \underline{I}^T) + \text{div}_S \underline{M} = \bar{O} \quad (23.27)$$

The equations of motion for the two-

dimensional body are thus equivalent

to (23.26) and (23.27).

The calculation of $\partial x (\bar{V}_0 \bar{T}^T)$ re-

quires $\bar{V}_0 \bar{T}$. We find that (see remark!)

$$\bar{V}_0 \bar{T} = \frac{\partial \bar{T}}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial \bar{T}}{\partial \theta_2} \dot{\theta}_2 = \bar{E}'_1 \dot{\theta}_1 + \bar{E}'_2 \dot{\theta}_2 \quad (23.28)$$

where the basis $(\bar{E}'_1, \bar{E}'_2, \bar{n})$ is recti-

procal to $(\bar{E}_1, \bar{E}_2, \bar{n})$. We denote this

linear mapping from $\bar{T}$ to the space

of 3-dim. vectors $(\bar{T} \in \mathbb{R}^3); \bar{I}$. Thus

$$\bar{I} : \bar{T} \rightarrow \bar{T}; \quad \bar{I} = \bar{E}'_1 \dot{\theta}_1 + \bar{E}'_2 \dot{\theta}_2 \quad (23.29)$$

We trivially have:

$$\bar{I} \bar{u} = \bar{u}, \quad \forall \bar{u} \in \bar{T} \quad (23.30)$$

We introduce the orthogonal projection

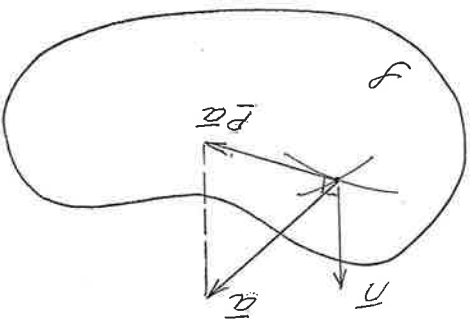
operator (on the tangent plane):

$$\bar{P} : \bar{T} \rightarrow \bar{T}; \quad \bar{I} \bar{P} = \bar{I} - \bar{n} \otimes \bar{n} \quad (23.31)$$

Here $\bar{I}$ is the unit tensor in $\bar{T} \cong \mathbb{R}^3$.

Note that:

$$\bar{P} \bar{u} = \bar{u}, \quad \forall \bar{u} \in \bar{T}; \quad \bar{P} \bar{n} = \bar{o} \quad (23.32)$$



$$\bar{u} \in \bar{T}, \quad \bar{a} \in \bar{T} \Rightarrow$$

$$\bar{u} \cdot \bar{P} \bar{a} = \bar{I} \bar{u} \cdot (\bar{a} - \bar{n} \cdot \bar{a} \bar{n}) = \bar{I} \bar{u} \cdot \bar{a} = \bar{u} \cdot \bar{P} \bar{a} = \bar{I}^T = \bar{P} \quad (23.33)$$

$$\bar{P} \bar{I} : \bar{T} \rightarrow \bar{T}; \quad \bar{u} \in \bar{T} \Rightarrow \bar{P} \bar{I} \bar{u} = \bar{u}$$

i.e.

$$\bar{P} \bar{I} = \bar{I}_S \quad (23.34)$$

where $\bar{I}_S$ is the unit tensor on $\bar{T}$.

$$\bar{I}_S \bar{u} = \bar{u}, \quad \forall \bar{u} \in \bar{T} \quad (23.35)$$

We thus have the simple relations:

$$\underline{I} \underline{P} = \underline{1} - \underline{n} \otimes \underline{n}, \quad \underline{P} \underline{I} = \underline{1}_s \quad (23.36)$$

Equation (23.27) may now be written:

$$\underline{H} - \text{cox}(\underline{I} \underline{I}^T) + \text{div}_s \underline{M} = \underline{0} \quad (23.37)$$

Now if we specialize to:

$$\underline{H} \equiv \underline{0}, \quad \underline{M} = \underline{0} \quad (\text{No moments!})$$

we get, from (23.27),

$$\text{cox}(\underline{I} \underline{I}^T) = \underline{0} \Leftrightarrow \underline{I} \underline{I}^T \text{ is symmetric}$$

$$\Leftrightarrow \underline{I} \underline{I}^T = (\underline{I} \underline{I}^T)^T = \underline{I}^T \underline{I} = \underline{I} \underline{P}$$

With the consequence:

$$\underline{u} \in \mathcal{F}_F \Rightarrow \underline{n} \cdot \underline{I} \underline{u} = \underline{I}^T \underline{n} \cdot \underline{u} =$$

$$\underline{I} \underline{I}^T \underline{n} \cdot \underline{u} = \underline{I} \underline{P} \underline{n} \cdot \underline{u} = 0 \Rightarrow$$

$$\underline{I} \underline{u} \in \mathcal{F}_F \quad (23.38)$$

i.e. $\underline{T} = \underline{I} \underline{U}$ is a tangent vector to the mean-surface! This should be compared to the situation for one-dimensional bodies where we obtained the eq. (see (21.14)!))

$$\underline{\bar{e}}_t \times \underline{T} + \frac{\partial \underline{M}}{\partial s} + \underline{h} = \underline{0}$$

and the implication (for the string):

$$\underline{h} \equiv \underline{0}, \quad \underline{M} \equiv \underline{0} \Rightarrow \underline{\bar{e}}_t \times \underline{T} = \underline{0}$$

i.e. $\underline{T}$ is tangent to the mean-curve!

If we now replace $\underline{I}$ with $\underline{P} \underline{I} = \underline{I}_s$

we get a symmetric tensor on the mean-surface: $\underline{I}_s; \mathcal{F}_F \rightarrow \mathcal{F}_F; \quad \underline{I}_s^T =$

$$\begin{aligned} (\underline{P} \underline{I})^T &= \underline{I}^T \underline{P}^T = \underline{I}^T \underline{I} = \underline{I}^T \underline{I} \underline{I}^T \underline{I} = \\ &= \underline{I}^T \underline{I} \underline{P} \underline{I} = \underline{P} \underline{I} = \underline{I}_s \\ &= \underline{1}_s \end{aligned} \quad (23.39)$$

Remark

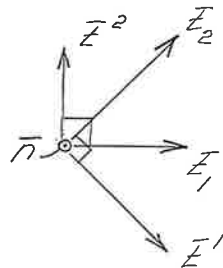
Some math. on surfaces (diff. geom.)

Let $\bar{a} = \bar{a}(\bar{r})$ be a vector field on S . We define the surface gradient of $\bar{a}$:

$$\nabla_s \bar{a} = \frac{\partial \bar{a}}{\partial \theta^1} \otimes \bar{E}^1 + \frac{\partial \bar{a}}{\partial \theta^2} \otimes \bar{E}^2 \quad (23.40)$$

where $(\bar{E}^1, \bar{E}^2)$ is the basis in $T_{\bar{r}}$ reciprocal to $(\bar{E}_1, \bar{E}_2)$:

$$\bar{E}^1 = \frac{\bar{E}_2 \times \bar{n}}{|\bar{E}_1 \times \bar{E}_2|}, \quad \bar{E}^2 = \frac{\bar{n} \times \bar{E}_1}{|\bar{E}_1 \times \bar{E}_2|}$$



Note that $\bar{E}_i \cdot \bar{E}^j = \delta_i^j$.

The tangential surface gradient of $\bar{a}$:

$$D\bar{a} = \underline{P} \nabla_s \bar{a} \quad (23.41)$$

This is a linear mapping $T_{\bar{r}} \rightarrow T_{\bar{r}}$.

The surface divergence of $\bar{a}$:

$$\text{div}_s \bar{a} = \text{tr}(D\bar{a}) = \frac{\partial \bar{a}}{\partial \theta^1} \cdot \bar{E}^1 + \frac{\partial \bar{a}}{\partial \theta^2} \cdot \bar{E}^2 \quad (23.42)$$

Let $\bar{n} = \bar{n}(\bar{r})$ be the unit normal vector field on S . Since $\bar{n} \cdot \bar{n} = 1$

$$\bar{n} \cdot \frac{\partial \bar{n}}{\partial \theta^1} = \bar{n} \cdot \frac{\partial \bar{n}}{\partial \theta^2} = 0$$

$$\nabla_s \bar{n} = \frac{\partial \bar{n}}{\partial \theta^1} \otimes \bar{E}^1 + \frac{\partial \bar{n}}{\partial \theta^2} \otimes \bar{E}^2$$

$$\bar{n} \cdot (\nabla_s \bar{n}) \bar{u} = 0, \quad \forall \bar{u} \in T_{\bar{r}} \Rightarrow$$

$$\underline{P}(\nabla_s \bar{n}) \bar{u} = (\nabla_s \bar{n}) \bar{u}, \quad \forall \bar{u} \in T_{\bar{r}}$$

The "Weingarten map" is defined by:

$$\underline{W} = -D\bar{n} \quad (23.43)$$

It follows that:

$$\underline{W} \bar{E}_1 = -\frac{\partial \bar{n}}{\partial \theta^1}, \quad \underline{W} \bar{E}_2 = -\frac{\partial \bar{n}}{\partial \theta^2}$$

$$\bar{n} \cdot \bar{E}_1 = 0 \Rightarrow \frac{\partial \bar{n}}{\partial \theta^2} \cdot \bar{E}_1 + \bar{n} \cdot \frac{\partial \bar{E}_1}{\partial \theta^2} = 0 \Rightarrow$$

$$\frac{\partial \bar{n}}{\partial \theta^2} \cdot \bar{E}_1 = -\bar{n} \cdot \frac{\partial^2 \bar{r}}{\partial \theta_2 \partial \theta_1}$$

$$\bar{n} \cdot \bar{E}_2 = 0 \Rightarrow \frac{\partial \bar{n}}{\partial \theta_1} \cdot \bar{E}_2 = -\bar{n} \cdot \frac{\partial^2 \bar{r}}{\partial \theta_1 \partial \theta_2}$$

and we have the relation

$$\left| \frac{\partial \bar{n}}{\partial \theta_2} \cdot \bar{E}_1 = \frac{\partial \bar{n}}{\partial \theta_1} \cdot \bar{E}_2 \Rightarrow \bar{E}_1 \cdot \underline{W} \bar{E}_2 = \bar{E}_2 \cdot \underline{W} \bar{E}_1 \right. \quad (23.44)$$

From this we may conclude:

$$\begin{aligned} \bar{u} \cdot \underline{W} \bar{v} &= (\bar{E}_1 u^1 + \bar{E}_2 u^2) \cdot \underline{W} (\bar{E}_1 v^1 + \bar{E}_2 v^2) = \\ &= \bar{E}_1 \cdot \underline{W} \bar{E}_1 u^1 v^1 + \bar{E}_2 \cdot \underline{W} \bar{E}_2 u^2 v^2 + u^1 v^2 \bar{E}_1 \cdot \underline{W} \bar{E}_2 + \\ &+ u^2 v^1 \bar{E}_2 \cdot \underline{W} \bar{E}_1 = \bar{v} \cdot \underline{W} \bar{u} \Leftrightarrow \underline{W}^T = \underline{W} \quad (23.45) \end{aligned}$$

where we have invoked (23.43). The Weingarten map is thus a symmetric tensor.

The Mean and Gaussian curvatures of $\mathcal{S}$ are given by:

$$\left| \kappa_m = \frac{1}{2} \operatorname{tr} \underline{W}, \quad \kappa_g = \det \underline{W} \right. \quad (23.46)$$

Let $\underline{A} = \underline{A}(\bar{r})$ be a tensor field on $\mathcal{S}$.

The surface gradient and surface divergence are defined by:

$$\left| \nabla_{\mathcal{S}} \underline{A} = \frac{\partial \underline{A}}{\partial \theta_1} \otimes \bar{E}^1 + \frac{\partial \underline{A}}{\partial \theta_2} \otimes \bar{E}^2 \right. \quad (23.47)$$

$$\operatorname{div}_{\mathcal{S}} \underline{A} = \frac{\partial \underline{A}}{\partial \theta_1} \bar{E}^1 + \frac{\partial \underline{A}}{\partial \theta_2} \bar{E}^2$$

The following calculation rules may be established:

$$\operatorname{div}_{\mathcal{S}} (f \bar{a}) = f \operatorname{div}_{\mathcal{S}} \bar{a} + \bar{a} \cdot \nabla_{\mathcal{S}} f$$

$$\operatorname{div}_{\mathcal{S}} (f \underline{A}) = f \operatorname{div}_{\mathcal{S}} \underline{A} + \underline{A} \nabla_{\mathcal{S}} f \quad (23.48)$$

$$\operatorname{div}_{\mathcal{S}} (\underline{A} \bar{a}) = (\operatorname{div}_{\mathcal{S}} \underline{A}) \cdot \bar{a} + \underline{A} \cdot \nabla_{\mathcal{S}} \bar{a}$$

where $f = f(\bar{r})$, $\bar{a} = \bar{a}(\bar{r})$ and $\underline{A} = \underline{A}(\bar{r})$ are scalar-, vector- and (2nd order) tensor fields respectively.

With

$$\bar{u} = \bar{u}_{\mathcal{S}} + (\bar{u} \cdot \bar{n}) \bar{n}, \quad \bar{u}_{\mathcal{S}} \cdot \bar{n} = 0$$

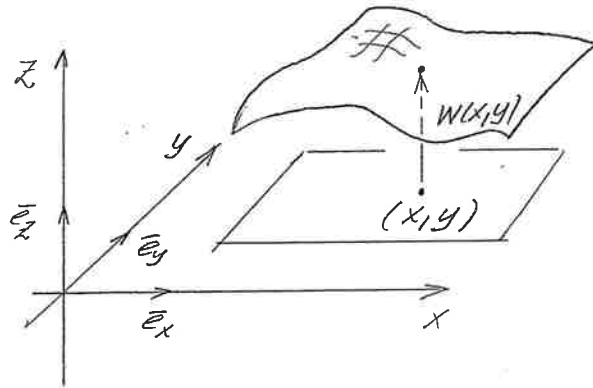
we find that

$$\operatorname{div}_{\mathcal{S}} \bar{u} = \operatorname{div}_{\mathcal{S}} \bar{u}_{\mathcal{S}} - 2\kappa_m \bar{n} \cdot \bar{u}$$

Try to prove this!

With the eq. of the surface on the form:

$$\vec{r} = \vec{r}(x, y) = \vec{e}_x x + \vec{e}_y y + \vec{e}_z W(x, y) \quad (23.49)$$



We have:

$$\begin{aligned} \vec{E}_x &= \vec{e}_x + \vec{e}_z h_x, & h_x &= \frac{\partial W}{\partial x} \\ \vec{E}_y &= \vec{e}_y + \vec{e}_z h_y, & h_y &= \frac{\partial W}{\partial y} \end{aligned} \quad (23.50)$$

$$\vec{n} = \frac{\vec{E}_x \times \vec{E}_y}{|\vec{E}_x \times \vec{E}_y|} = \frac{\vec{e}_x(-h_x) + \vec{e}_y(-h_y) + \vec{e}_z}{\sqrt{1 + h_x^2 + h_y^2}}$$

The mean curvature:

$$\kappa_m = \frac{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}}{2(1 + h_x^2 + h_y^2)^{3/2}} \quad (23.51)$$

The Gaussian curvature:

$$\kappa_g = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2} \quad (23.52)$$

where

$$h_{xx} = \frac{\partial^2 W}{\partial x^2}, \quad h_{yy} = \frac{\partial^2 W}{\partial y^2}, \quad h_{xy} = \frac{\partial^2 W}{\partial x \partial y} \quad (23.53)$$

Note that if $h_x^2 + h_y^2 \ll 1$ then

$$\kappa_m \approx \frac{1}{2}(h_{yy} - 2h_x h_y h_{xy} + h_{xx}) \quad (23.54)$$

$$\kappa_g \approx h_{xx}h_{yy} - h_{xy}^2 \quad (23.55)$$

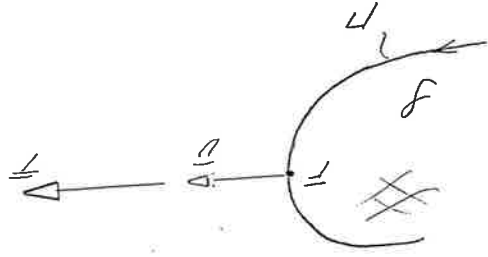
If we drop the mid-term in (23.54) then:

$$\kappa_m \approx \frac{1}{2} \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} \right) = \frac{1}{2} \Delta W \quad (23.56)$$

#

LECTURE 18

CONTINUOUS SYSTEMS
The two-dimensional body
The membrane
Thin plates



The membrane
This is the two-dimensional analog
of the previously encountered one-di-
mensional string. We here assume that:

$$H = 0, \quad M = 0 \quad (23.57)$$

and that (1 unit tensor: $\mathbf{I}_0 \mathbf{v} = \mathbf{v}$)

$$\bar{\mathbf{T}}_0 = \mathbf{T}_0 \mathbf{I}_0, \quad \mathbf{T}_0 = \mathbf{T}_0(\mathbf{f}, \mathbf{f}) \quad (23.58)$$

i.e. we have a "surface tension

"state" in the membrane defined by
the surface tension $\mathbf{T}_0$. Then the

fraction vector may be written:

$$\bar{\mathbf{T}} = \bar{\mathbf{T}} \mathbf{v} = \mathbf{T}_0 \mathbf{I}_0 \mathbf{v} = \mathbf{T}_0 \mathbf{v} \in \mathcal{J}_f^* \quad (23.59)$$

for $\mathbf{v} \in \mathcal{J}_f^*$. $\bar{\mathbf{T}}$ is thus parallel to $\mathbf{v}$.

The eq. (23.37) is now identically satisfied. Equation (23.26) implies:

$$\bar{K} + \operatorname{div}_S (T_0 \underline{1}_S) = \bar{a} m \quad (23.60)$$

But from the calculation rules (23.48) we have: ($\kappa_m = \frac{1}{2} \operatorname{tr}(\underline{W})$)

$$\operatorname{div}_S (T_0 \underline{1}_S) = \nabla_S T_0 + 2\kappa_m T_0 \bar{n} \quad (23.61)$$

and consequently:

$$\bar{K} + \nabla_S T_0 + 2\kappa_m T_0 \bar{n} = \bar{a} m \quad (23.62)$$

If now

$$\bar{K} = \bar{n} p$$

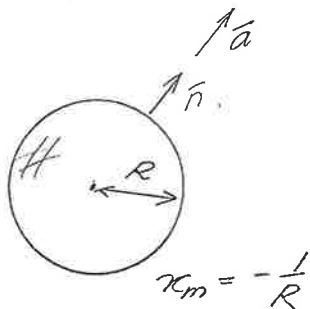
and

$$\bar{a} = \bar{n} a$$

then

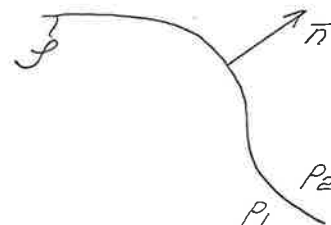
$$\nabla_S T_0 = 0 \Leftrightarrow T_0 = T_0(t) \quad (23.63)$$

$$p + 2\kappa_m T_0 = a m$$

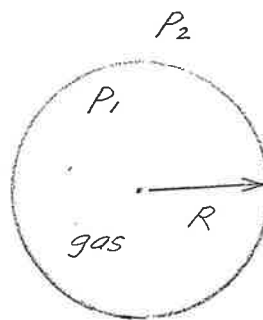


In a static situation ($a=0$) with gas - pressures p_1 and p_2 respectively on each side of the membrane we get Laplace's formula:

$$p_1 - p_2 + 2\kappa_m T_0 = 0 \quad (23.64)$$



Example ("The Balloon")



$$p = p_1 - p_2$$

$$\kappa_m = -\frac{1}{R}$$

According to (23.63)

$$p + 2\left(-\frac{1}{R}\right) T_0 = a m \quad (23.65)$$

First we look at an equilibrium configuration, i.e. $Q=0$, $R=R_0$, $p=p_0$. Then

$$p_0 + 2\left(-\frac{1}{R_0}\right)T_0 = 0 \Rightarrow$$

$$p_0 = \frac{2T_0}{R_0} \Leftrightarrow T_0 = \frac{p_0 R_0}{2} \quad (23.66)$$

Assuming constant temperature and an ideal gas

$$pV = p_0 V_0 \Rightarrow p = p_0 \left(\frac{R_0}{R}\right)^3 = \frac{2T_0}{R_0} \left(\frac{R_0}{R}\right)^3 \quad (23.67)$$

Inserting this into (23.65)

$$\frac{2T_0}{R_0} \left(\frac{R_0}{R}\right)^3 - \frac{2T_0}{R} = am = \ddot{R}m \quad (23.68)$$

Taking

$$m = m_0 \left(\frac{R_0}{R}\right)^2 \quad (23.69)$$

we obtain

$$\frac{2T_0}{m_0 R_0^2 R^2} (R_0^3 - R^3) = \ddot{R} \quad (23.70)$$

or

$$\frac{2T_0}{m_0 R_0^2 R^2} (R_0^2 + R_0 R + R^2)(R_0 - R) = \ddot{R} \quad (23.71)$$

Taking $r = R - R_0$ and linearizing (23.71) at $R = R_0$ gives the differential equation for small radial vibrations of the balloon around its equilibrium configuration

$$\ddot{r} + \frac{2T_0}{m_0 R_0^2 R_0^2} (R_0^2 + R_0 R_0 + R_0^2)r = 0 \quad (23.72)$$

or

$$\ddot{r} + \omega^2 r = 0 \quad (23.73)$$

where the angular frequency of the vibrations is given by

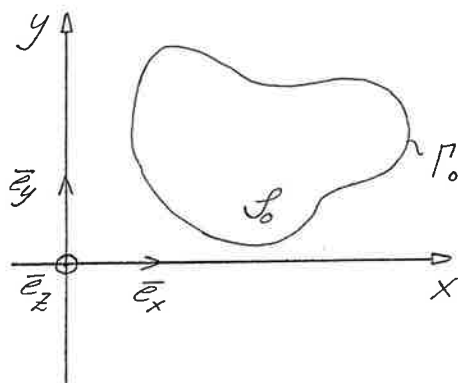
$$\omega^2 = \frac{6T_0}{m_0 R_0^2} = \frac{24\pi T_0}{\mu_b} = \frac{12\pi p_0 R_0}{\mu_b} \quad (23.74)$$

320d

where $\mu_b = 4\pi R_b^2 m_b$ is the mass of the
balloon.

#

Now let the membrane in the (undeformed) referential configuration be a plane surface in the x - y -plane.



The displacement of material points on the mid-surface:

$$\bar{u} = \bar{u}(x, y, t)$$

and then

$$\bar{r} = \bar{r}(x, y, t) = \bar{e}_x x + \bar{e}_y y + \bar{u}(x, y, t)$$

$$\bar{t}_x = \frac{\partial \bar{r}}{\partial x} = \bar{e}_x + \frac{\partial \bar{u}}{\partial x}$$

$$\bar{t}_y = \frac{\partial \bar{r}}{\partial y} = \bar{e}_y + \frac{\partial \bar{u}}{\partial y}$$

tangent vectors

We now introduce the simplifying assumptions:

$$\bar{u}(x, y, t) = \bar{e}_z w(x, y, t) \quad (23.75)$$

$$\bar{K} = p \bar{e}_z$$

Then

$$\bar{a} = \ddot{w} \bar{e}_z \quad (23.76)$$

and from (23.62) we get

$$p + \bar{e}_z \cdot \nabla_s T_0 + 2 \kappa_m T_0 \bar{e}_z \cdot \bar{n} = \ddot{w} m \quad (23.77)$$

where we now use the approximations:

$$\kappa_m \cong \frac{1}{2} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \frac{1}{2} \Delta w \quad (23.78)$$

$$\bar{e}_z \cdot \bar{n} = \frac{1}{\sqrt{1 + \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2}} \cong 1$$

$$\text{Then } \bar{e}_z \cdot \nabla_s T_0 \cong \bar{n} \cdot \nabla_s T_0 = 0$$

Furthermore if

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 \ll 1 \quad (23.79)$$

then, we obtain the membrane equation

$$p + T_0 \Delta W = \ddot{W} m \quad (23.80)$$

We also have the equations:

$$\bar{e}_x \cdot \nabla_s T_0 + 2\kappa_m T_0 \bar{e}_x \cdot \bar{n} = 0 \quad (23.81)$$

$$\bar{e}_y \cdot \nabla_s T_0 + 2\kappa_m T_0 \bar{e}_y \cdot \bar{n} = 0$$

where

$$\bar{e}_x \cdot \bar{n} \approx -\frac{\partial W}{\partial x}, \quad \bar{e}_y \cdot \bar{n} \approx -\frac{\partial W}{\partial y}$$

Note that

$$\nabla_s T_0 = 0 \Leftrightarrow T_0 = T_0(z)$$

then from (23.81)

$$\left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)\left(\frac{\partial W}{\partial x}\right) = \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right)\left(\frac{\partial W}{\partial y}\right) = 0$$

This condition will of course only be approximately satisfied and it requires small displacements W of the membrane.

A simple boundary condition in connection with (23.80) is the clamped boundary:

$$W(x, y, z) = 0, \quad (x, y) \in \Gamma_0 \quad (23.82)$$

Note that a free part of the boundary is not compatible with the assumption:

$$T_0 = \text{constant}$$

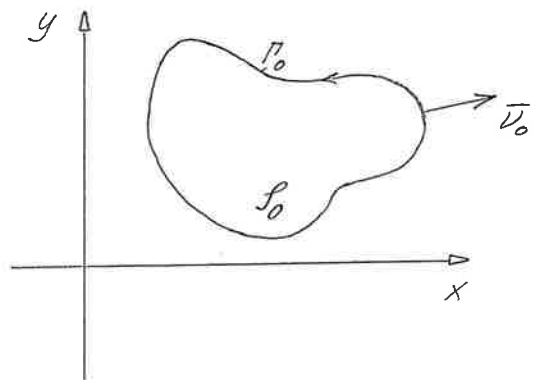
If T_0 is not constant and not known beforehand we need another equation, relating T_0 to the deformation of the membrane ("a constitutive relation").

The thin plate

For a thin plate we here assume that

$$\bar{H} \equiv \bar{O} \quad (23.83)$$

Let $S_0 \subseteq \mathbb{R}^2$ be the plate in the undeformed referential position,



and let

$$\bar{u} = \bar{u}(x, y, t)$$

be the displacement of the mid-surface.

We introduce the traction and moment

tensors, $\bar{\Pi}_0$ and $\bar{M}_0$ respectively,

defined by the relations:

$$\int_{\Gamma_0} \bar{\Pi}_0 \bar{V}_0 dS_0 = \int_{\Gamma} \bar{\Pi} \bar{V} dS \quad (23.84)$$

$$\int_{\Gamma_0} \bar{M}_0 \bar{V}_0 dS_0 = \int_{\Gamma} \bar{M} \bar{V} dS$$

Then the equations of motion may be written:

$$\bar{K}_0 + \text{div}_{S_0} \bar{\Pi}_0 = \bar{a} m_0 \quad (23.85)$$

$$-2ax(\nabla_{S_0} \bar{F} \bar{\Pi}_0^T) + \text{div}_{S_0} \bar{M}_0 = \bar{O}$$

where now $\bar{K}_0$ and m_0 are measured per unit area of S_0 !

Note that:

$$\nabla_{S_0} \bar{F} = \frac{\partial \bar{F}}{\partial x} \otimes \bar{e}_x + \frac{\partial \bar{F}}{\partial y} \otimes \bar{e}_y = \quad (23.86)$$

$$(\bar{e}_x + \frac{\partial \bar{u}}{\partial x}) \otimes \bar{e}_x + (\bar{e}_y + \frac{\partial \bar{u}}{\partial y}) \otimes \bar{e}_y$$

For the thin plate we now assume that:

$$\bar{u}(x, y, t) = \bar{e}_z w(x, y, t), \quad \bar{a} = \bar{e}_z \ddot{w} \quad (23.87)$$

$$\bar{\pi}_0 = \bar{e}_z \otimes \bar{S}$$

where

$$\bar{S} = \bar{e}_x S_x + \bar{e}_y S_y \quad (23.88)$$

is the shear vector. The traction vector:

$$\bar{T}_0 = \bar{\pi}_0 \bar{V}_0 \quad (\bar{T} ds = \bar{T}_0 ds_0)$$

$$\Rightarrow \bar{T}_0 = (\bar{e}_z \otimes \bar{S}) \bar{V}_0 = \bar{e}_z (\bar{S} \cdot \bar{V}_0)$$

which means that we are only considering "shear forces" acting across Γ_0 !

We take:

$$\begin{aligned} \bar{M}_0 = & -M_{xy} \bar{e}_x \otimes \bar{e}_x - M_y \bar{e}_x \otimes \bar{e}_y + \\ & M_x \bar{e}_y \otimes \bar{e}_x + M_{xy} \bar{e}_y \otimes \bar{e}_y \end{aligned} \quad (23.89)$$

which means that the matrix of

$\bar{M}_0$ relative to the basis $(\bar{e}_x, \bar{e}_y)$ is equal to:

$$\begin{pmatrix} -M_{xy} & -M_y \\ M_x & M_{xy} \end{pmatrix}$$

Note that: (and see figure below!)

$$\bar{M}_0 \bar{e}_x = \bar{e}_x (-M_{xy}) + \bar{e}_y M_x \quad (23.90)$$

$$\bar{M}_0 \bar{e}_y = \bar{e}_x (-M_y) + \bar{e}_y M_{xy}$$

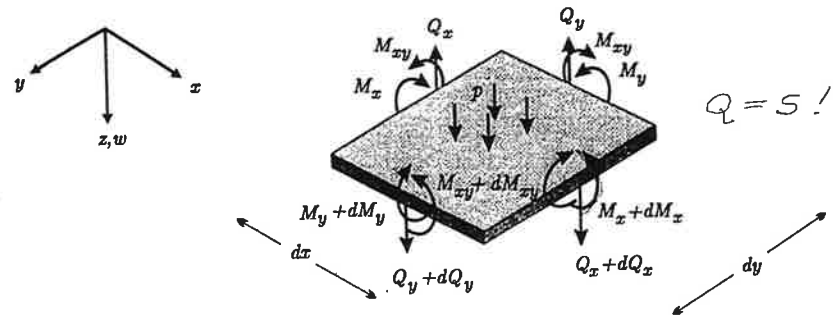


Figure 4.6.6
Equilibrium of the plate element

M_x, M_y : bending moments
 M_{xy} : twisting moment (torsion)

We have:

$$\operatorname{div}_{\rho_0} \underline{\Pi}_0 = \frac{\partial \underline{\Pi}_0}{\partial x} \bar{e}_x + \frac{\partial \underline{\Pi}_0}{\partial y} \bar{e}_y = \bar{e}_z \left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} \right). \quad (23.91)$$

$$\begin{aligned} \nabla_{\rho_0} \underline{F} \underline{\Pi}_0^T &= \left\{ (\bar{e}_x + \bar{e}_z \frac{\partial W}{\partial x}) \otimes \bar{e}_x \right. \\ &\quad \left. (\bar{e}_y + \bar{e}_z \frac{\partial W}{\partial y}) \otimes \bar{e}_y \right\} \bar{S} \otimes \bar{e}_z \Big\} = \\ &= S_x \bar{e}_x \otimes \bar{e}_z + S_y \bar{e}_y \otimes \bar{e}_z + \bar{e}_z \otimes \bar{e}_z \left(S_y \frac{\partial W}{\partial x} + \right. \\ &\quad \left. S_x \frac{\partial W}{\partial y} \right) \Rightarrow (\operatorname{ox}(\bar{e}_x \otimes \bar{e}_z) = -\frac{1}{2} \bar{e}_x \times \bar{e}_z = \frac{1}{2} \bar{e}_y) \\ &\Rightarrow \\ \operatorname{ox}(\nabla_{\rho_0} \underline{F} \underline{\Pi}_0^T) &= S_x \operatorname{ox}(\bar{e}_x \otimes \bar{e}_z) + S_y \operatorname{ox}(\bar{e}_y \otimes \bar{e}_z) = \\ &= \bar{e}_x \left(-\frac{1}{2} S_y \right) + \bar{e}_y \left(\frac{1}{2} S_x \right) \end{aligned} \quad (23.92)$$

$$\begin{aligned} \operatorname{div}_{\rho_0} \underline{M}_0 &= \frac{\partial \underline{M}_0}{\partial x} \bar{e}_x + \frac{\partial \underline{M}_0}{\partial y} \bar{e}_y = \\ &= \bar{e}_x \left(-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} \right) + \bar{e}_y \left(\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} \right) \end{aligned} \quad (23.93)$$

These relations inserted into

the equations (23.75)₂ with the external force:

$$\underline{\bar{F}}_0 = \rho_0 \bar{e}_z \quad (23.94)$$

gives

$$\begin{aligned} \rho_0 + \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} &= \ddot{w} m_0 \\ S_y - \frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} &= 0 \\ -S_x + \frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x} &= 0 \end{aligned} \quad (23.95)$$

The last two equations may be written:

$$\begin{aligned} S_x &= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \\ S_y &= \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \end{aligned} \quad (23.96)$$

These relations should be compared with the corresponding beam-relation (21.41).

Now we may substitute (23.96) into (23.95), with the result:

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + p_0 = m_0 \ddot{w} \quad (23.97)$$

"The constitutive equations" for a

linear elastic plate are:

$$M_x = -D \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = -D \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right)$$

$$M_{xy} = -D(1-\nu) \frac{\partial^2 w}{\partial x \partial y}$$

where the plate bending stiffness:

$$D = \frac{E h^3}{12(1-\nu^2)} \quad (23.99)$$

E is the modulus of elasticity and ν the Poisson number.

The equations (23.98) may be derived from first principles in the theory

of elasticity:

Substitution of (23.98) into (23.97) gives

the equation of plate vibrations:

$$D \Delta^2 w - p_0 + m_0 \ddot{w} = 0 \quad (23.100)$$

where the "double Laplacian" *

$$\Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2}$$

(*biharmonic operator)

For a simply supported plate we

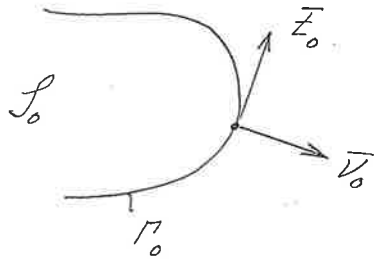
have the boundary conditions:

$$w = 0$$

$$\bar{E}_0 \cdot \bar{M}_0 = (\bar{e}_x(-v_y) + \bar{e}_y v_x) \cdot (\bar{e}_x(-M_{xy} v_x -$$

$$M_y v_y) + \bar{e}_y (M_x v_x + M_{xy} v_y) =$$

$$M_x v_x^2 + M_y v_y^2 + 2 M_{xy} v_x v_y = 0, \text{ on } \Gamma_0$$



For the damped plate we have:

$$\begin{aligned} & W = 0 \\ & \bar{n} \cdot \bar{D}_0 = \nabla W \cdot \bar{D}_0 = \frac{\partial W}{\partial x} v_x + \frac{\partial W}{\partial y} v_y = 0 \end{aligned} \quad (23.92)$$

Note that

$$\bar{n} \cdot \bar{E}_0 = \nabla W \cdot \bar{E}_0 = 0$$

since $W = 0$ on Γ_0 .

LECTURE 19

CONTINUOUS SYSTEMS
AND APPROXIMATIVE METHODS
The Rayleigh – Ritz method

24. Weak and variational formulations
of the vibration problem.

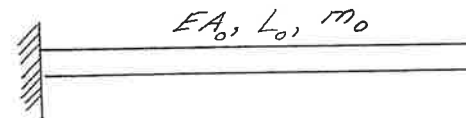
We look at longitudinal vibrations of the bar. The differential equation governing the motion is:

$$\left\| \frac{d}{dx} \left(EA_0 \frac{\partial u}{\partial x} \right) + k_0 = \ddot{u} m_0 \right. \quad (24.1)$$

$$u = u(x, t), \quad 0 < x < L_0, \quad t \geq 0$$

We take the clamped-free bar with boundary conditions: $(N = EA_0 \frac{\partial u}{\partial x})$

$$\left\| \begin{aligned} u(0, t) &= 0 \\ \frac{\partial u}{\partial x}(L_0, t) &= 0 \quad (\Leftrightarrow N(L_0, t) = 0) \end{aligned} \right. \quad (24.2)$$



and no external force

$$\left\| k_0(x, t) \equiv 0 \right. \quad (24.3)$$

We look for standing wave solutions

$$u(x, t) = \hat{u}(x) \sin \omega t$$

Then ("Strong formulation")

$$\frac{d}{dx} \left(EA_0 \frac{d\hat{u}}{dx} \right) + \omega^2 m_0 \hat{u} = 0$$

$$\hat{u}(0) = 0 \quad (24.4)$$

$$\frac{d\hat{u}}{dx}(L_0) = 0$$

We have seen that this system has solutions:

$$\begin{aligned} \hat{u}_1, \hat{u}_2, \hat{u}_3, \dots \\ \omega_1, \omega_2, \omega_3, \dots \rightarrow \infty \end{aligned} \quad (24.5)$$

representing mode shapes and corresponding eigenfrequencies. It may be

shown that $\hat{u}_i$ and $\hat{u}_j$ are orthogonal in the sense ($EA_0 = \text{constant}$):

$$\int_0^{L_0} \hat{u}_i(x) \hat{u}_j(x) dx = \delta_{ij} \quad (24.6)$$

We introduce the following function spaces related to problem (24.4).

The solution space:

$$S = \{ \hat{w} = \hat{w}(x) \in C^2[0, L_0], \hat{w}(0) = 0, \hat{w}'(L_0) = 0 \} \quad (24.7)$$

The essential space (the space of variations):

$$E = \{ \hat{w} = \hat{w}(x) \in C^1[0, L_0], \hat{w}(0) = 0 \} \quad (24.8)$$

The boundary condition

$$\hat{w}(0) = 0 \quad (24.9)$$

is said to be of the essential type. It represents a geometric constraint on the displacements. The boundary condition

$$N(L_0, t) = 0 \Leftrightarrow \frac{d\hat{u}}{dx}(L_0) = 0 \quad (24.10)$$

is of the natural type representing conditions on external forces.

obviously both spaces S and $\tilde{E}$ are linear, in this case, and $S \subseteq \tilde{E}$.

The scalar product $\langle \varphi, \psi \rangle$ is defined by:

$$\langle \varphi, \psi \rangle = \int_0^{L_0} \varphi(x) \psi(x) dx \quad (24.11)$$

We introduce the potential energy functional: $V: \tilde{E} \rightarrow \mathbb{R}$ defined by

$$V(\hat{w}) = \frac{1}{2} EA_0 \langle \hat{w}', \hat{w}' \rangle - \frac{1}{2} \omega^2 m_0 \langle \hat{w}, \hat{w} \rangle \quad (24.12)$$

and the stationarity formulation:

Find $\hat{u} \in \tilde{E}$ such that for the function $g(\varepsilon) = V(\hat{u} + \varepsilon \hat{v})$, $\varepsilon \in \mathbb{R}$

$$g'(0) = 0, \quad \forall \hat{v} \in \tilde{E} \quad (24.13)$$

and the variational ("weak") formulation:

Find $\hat{u} \in \tilde{E}$ such that

$$EA \langle \hat{u}', \hat{w}' \rangle - \omega^2 m_0 \langle \hat{u}, \hat{w} \rangle = 0, \quad \forall \hat{w} \in \tilde{E} \quad (24.14)$$

Strong $\rightarrow$ weak: We now show that if $\hat{u}$ is a solution to (24.4) then $\hat{u}$ is

is a solution to the variational problem (24.13): From (24.4), we find that

$$EA_0 \langle \hat{u}'', \hat{w} \rangle + \omega^2 m_0 \langle \hat{u}, \hat{w} \rangle = 0, \quad \forall \hat{w} \in \tilde{E}$$

and $\hat{u} \in S$. But, using partial integration

$$\begin{aligned} \langle \hat{u}'', \hat{w} \rangle &= \int_0^{L_0} \hat{u}'' \hat{w} dx = [\hat{u}' \hat{w}]_0^{L_0} - \\ &\int_0^{L_0} \hat{u}' \hat{w}' dx = \underbrace{(\hat{u}'(L_0) \hat{w}(L_0))}_{=0} - \underbrace{\hat{u}'(0) \hat{w}(0)}_{=0} - \end{aligned}$$

$$\int_0^{L_0} \hat{u}' \hat{w}' dx = - \langle \hat{u}', \hat{w}' \rangle, \quad \forall \hat{w} \in \tilde{E}$$

where we have used the boundary condition (24.3)₃. Then

$$-EA_0 \langle \hat{u}', \hat{w}' \rangle + \omega^2 m_0 \langle \hat{u}, \hat{w} \rangle = 0, \quad \forall \hat{w} \in \tilde{E}$$

Weak $\rightarrow$ stationarity: Now assume that $\hat{u}$ is a solution to the variational problem, then

$$\begin{aligned} g(\varepsilon) &= V(\hat{u} + \varepsilon \hat{v}) = \frac{1}{2} EA_0 \langle \hat{u}' + \varepsilon \hat{v}', \hat{u}' + \varepsilon \hat{v}' \rangle - \\ &\frac{1}{2} \omega^2 m_0 \langle \hat{u} + \varepsilon \hat{v}, \hat{u} + \varepsilon \hat{v} \rangle = \frac{1}{2} EA_0 \langle \hat{u}', \hat{u}' \rangle - \end{aligned}$$

$$- \frac{1}{2} \omega^2 m_0 \langle \hat{u}, \hat{u} \rangle + E(EA \langle \hat{u}', \hat{v}' \rangle - \omega^2 m_0 \langle \hat{u}, \hat{v} \rangle) + E^2 \left(\frac{1}{2} EA \langle \hat{v}', \hat{v}' \rangle - \frac{1}{2} \omega^2 m_0 \langle \hat{v}, \hat{v} \rangle \right) = 0$$

$$g'(0) = EA \langle \hat{u}', \hat{v}' \rangle - \omega^2 m_0 \langle \hat{u}, \hat{v} \rangle = 0$$

$\forall \hat{v} \in \mathcal{E}$ which means that $\hat{u}$ is a solution to the stationarity problem. The converse is obviously also true.

Variational $\rightarrow$ strong: Finally we note that if $\hat{u}$ is a solution to the variational problem, (24.12), and if $\hat{u} \in S$ then:

$$EA \langle \hat{u}', \hat{w}' \rangle - \omega^2 m_0 \langle \hat{u}, \hat{w} \rangle = - \int_0^L (EA \hat{u}'' + \omega^2 m_0 \hat{u}) \hat{w} dx = 0, \quad \forall \hat{w} \in \mathcal{E} \quad (24.16)$$

where we have used partial integration and the boundary conditions. From (24.16) it follows that:

$$EA \hat{u}'' + \omega^2 m_0 \hat{u} = 0$$

What will happen with a different boundary condition at the free end of the bar? If we for instance take

$$N(L) = EA \frac{d\hat{u}}{dx}(L) = N_0 \quad (24.17)$$

where N_0 is a given constant. Then from (24.4) it follows:

$$-EA \langle \hat{u}', \hat{w}' \rangle + N_0 \hat{w}(L) + \omega^2 m_0 \langle \hat{u}, \hat{w} \rangle = 0 \quad \forall \hat{w} \in \mathcal{E}. \quad (24.18)$$

defining the variational problem. The corresponding potential energy functional may be written:

$$V(\hat{w}) = \frac{1}{2} EA \langle \hat{w}', \hat{w}' \rangle - \frac{1}{2} \omega^2 m_0 \langle \hat{w}, \hat{w} \rangle + N_0 \hat{w}(L) \quad (24.19)$$

The Rayleigh-Ritz method

Let

$$f_1, f_2, \dots, f_N, \dots \quad (24.20)$$

be a set of linearly independent functions belonging to $\mathcal{B}$. Mathematically

we think of (24.20) as a set of basis-

functions in $\mathcal{B}$, i.e. $f_i \in C^1[0, L]$, $f_i(0) = 0$.

Let us seek an approximation $\hat{u}_N$ on

the form:

$$\hat{u}_N = \sum_{i=1}^N q_i f_i, \quad q_i \in \mathbb{R} \quad (24.21)$$

to the solution of the stationarity

problem (24.12). "The approximate

potential":

$$V_N(q_1, \dots, q_N) = V(\hat{u}_N) =$$

$$\frac{1}{2} \sum_{i,j=1}^N EA \langle f_i', f_j' \rangle q_i q_j - \frac{1}{2} \omega^2 \sum_{i,j=1}^N m_0 \langle f_i, f_j \rangle q_i q_j. \quad (24.22)$$

or in matrix-format:

$$V_N(q) = \frac{1}{2} q^T K q - \frac{1}{2} \omega^2 q^T M q \quad (24.23)$$

where

$$q = [q_1, q_2, \dots, q_N]^T$$

$$K = [k_{ij}], \quad k_{ij} = EA \langle f_i', f_j' \rangle \quad (24.24)$$

$$M = [m_{ij}], \quad m_{ij} = m_0 \langle f_i, f_j \rangle$$

Note that

$$k_{ij} = EA \int_0^L f_i'(x) f_j'(x) dx = k_{ji}, \quad K^T = K \quad (24.25)$$

$$m_{ij} = m_0 \int_0^L f_i(x) f_j(x) dx = m_{ji}, \quad M^T = M$$

For $\hat{q} = q$ to be a stationary point of $V_N = V_N(q)$:

$$\frac{\partial V_N}{\partial q_i}(q) = 0 \quad i = 1, 2, \dots, N \quad (24.26)$$

This is equivalent to:

$$(-\omega^2 M + K) \hat{q} = 0 \quad (24.27)$$

an equation we are familiar with from the theory of undamped finite dimensional vibration problem. $\underline{M}$ is of course called the mass matrix and $\underline{K}$ the stiffness matrix of the Rayleigh-Ritz approximation. Both matrices are of the $N \times N$ -type.

EX 17 The clamped-free bar.

We take (as basis):

$$\| f_1(x) = \frac{x}{L}, \quad f_2(x) = \left(\frac{x}{L}\right)^2 \quad (24.28)$$

and compute the mass-matrix:

$$m_{11} = \int_0^L m_0 f_1(x) f_1(x) dx = \int_0^L m_0 \left(\frac{x}{L}\right)^2 dx = \frac{m_0 L}{3}$$

$$m_{12} = m_{21} = \int_0^L m_0 \left(\frac{x}{L}\right)^3 dx = \frac{m_0 L}{4} \quad (24.29)$$

$$m_{22} = \int_0^L m_0 \left(\frac{x}{L}\right)^4 dx = \frac{m_0 L}{5}$$

$$\underline{M} = m_0 L \begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{pmatrix} \quad (24.30)$$

and the stiffness matrix:

$$k_{11} = \int_0^L EA (f_1')^2 dx = \int_0^L EA \frac{1}{L^2} dx = \frac{EA}{L}$$

$$k_{12} = k_{21} = \int_0^L EA \frac{2x}{L^3} dx = \frac{EA}{L} \quad (24.31)$$

$$k_{22} = \int_0^L EA \frac{4x^2}{L^4} dx = \frac{4}{3} \frac{EA}{L}$$

$$\underline{K} = \frac{EA}{L} \begin{pmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{pmatrix}$$

The eigenvalue problem:

$$(-\omega^2 \underline{M} + \underline{K}) \hat{q} = \vec{0}, \quad \hat{q} \neq 0 \quad (24.32)$$

$$\Rightarrow \det(-\omega^2 \underline{M} + \underline{K}) = 0 \quad \omega \neq 0$$

$$3\mu^2 - 104\mu + 240 = 0$$

Where

$$\mu = \omega^2 \frac{m_0 L^2}{EA}$$

Roots:

$$\mu_1 = 2.49 \Rightarrow \omega_1^2 = 2.49 \frac{EA}{m_0 L^2} \quad (24.33)$$

$$\mu_2 = 32.18 \Rightarrow \omega_2^2 = 32.18 \frac{EA}{m_0 L^2}$$

If we compare this with "the exact eigenfrequencies" ($\omega_n^2 = (2n-1)^2 \frac{\pi^2}{4} \frac{EA}{m_0 L^2}$)

$$\omega_1^2 = 2.467 \frac{EA}{m_0 L^2}, \quad \omega_2^2 = 22.21 \frac{EA}{m_0 L^2}$$

We find that the first eigentre-
quency is pretty well approximated
while the second is not. If we add
a third function to the basis:

$$f_3(x) = \left(\frac{x}{L}\right)^3 \quad (24.34)$$

then we get the following eigen-

frequencies:

	approx.	exact
ω_1^2	$2.47 \frac{EA}{m_0 L^2}$	$2.467 \frac{EA}{m_0 L^2}$
ω_2^2	$23.4 \frac{EA}{m_0 L^2}$	$22.21 \frac{EA}{m_0 L^2}$
ω_3^2	$109.2 \frac{EA}{m_0 L^2}$	$61.68 \frac{EA}{m_0 L^2}$

The conclusion is that the number
of base functions needed is at least
one more than the number of eigen-
frequencies to be computed.

#

The basis functions (24.20) are often
chosen as polynomials. It is possible
to select basis functions that are or-
thogonal as well (using Gram-Schmidt
orthonormalization process.)

$$\int_0^L f_i(x) f_j(x) dx = L \delta_{ij}, \quad 1 \leq i, j \leq N \quad (24.35)$$

and we have

$$\underline{M} = m_0 \underline{1} \quad (24.36)$$

However it does not automatically follow that:

$$\int_0^L f_i'(x) f_j'(x) dx = \frac{1}{L} \delta_{ij} \quad (24.37)$$

which means that the stiffness matrix may be far from diagonal. This is a major disadvantage with the Rayleigh-Ritz method. Another (however minor obstacle) is that the coordinates used, $\bar{q}$, do not have a simple physical interpretation. Note that the displacement at x ; $\hat{u}(x)$ will in general depend on all the q_i 's. These coordinates are, so to speak, "global".

LECTURE 20

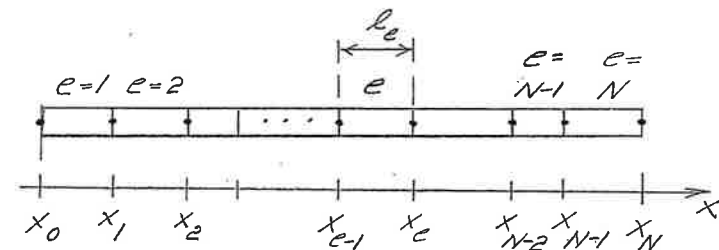
CONTINUOUS SYSTEMS
AND APPROXIMATIVE METHODS

The Finite element method

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The finite element method.

The finite element method may be considered as a special application of the Rayleigh-Ritz method where the basis functions are selected in a way that automatically results in almost diagonal (or banded) stiffness- and mass-matrices. We once again consider our model problem: the vibrating bar. We subdivide the bar into a number of finite parts ("finite elements") according to the figure below.



The element $n \equiv "e"$:

$$\{x \in \mathbb{R} : x_{e-1} \leq x \leq x_e\}, EA, m_0 \quad (24.39)$$

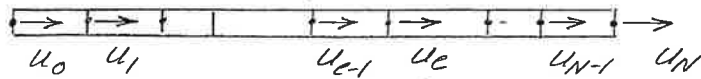
The nodes:

$$x_0, x_1, \dots, x_N \quad (24.39)$$

where x_{e-1} and x_e are "belonging to" element "e".

Nodal displacements are given by

$$u_0, u_1, \dots, u_N \quad (24.40)$$



For the element displacement we assume

$$\hat{u}_e(x) = u_{e-1} \phi_{e1}(x) + u_e \phi_{e2}(x), \quad x_{e-1} \leq x \leq x_e \quad (24.41)$$

where we now choose the functions

ϕ_{e1} and ϕ_{e2} so that:

$$\hat{u}_e(x_{e-1}) = u_{e-1}, \quad \hat{u}_e(x_e) = u_e \quad (24.42)$$

If we take ϕ_{e1} and ϕ_{e2} as linear

$$\phi_{e1}(x) = c_{11}^e + c_{12}^e x, \quad \phi_{e2}(x) = c_{21}^e + c_{22}^e x \quad (24.43)$$

and use the fact that

$$\phi_{e1}(x_{e-1}) = 1, \quad \phi_{e1}(x_e) = 0 \quad (24.44)$$

$$\phi_{e2}(x_{e-1}) = 0, \quad \phi_{e2}(x_e) = 1$$

we find that:

$$\phi_{e1}(x) = \chi_1\left(\frac{x - x_{e-1}}{l_e}\right), \quad x_{e-1} \leq x \leq x_e \quad (24.45)$$

$$\phi_{e2}(x) = \chi_2\left(\frac{x - x_{e-1}}{l_e}\right), \quad x_{e-1} \leq x \leq x_e$$

where l_e is the element length

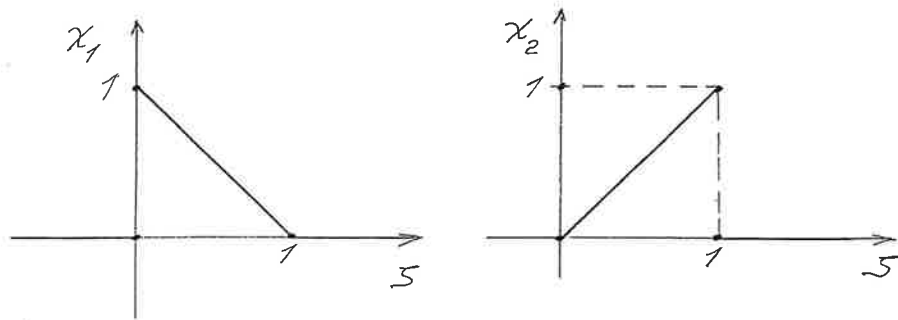
$$l_e = x_e - x_{e-1} \quad e = 1, 2, \dots, N \quad (24.46)$$

The so-called shape functions χ_1 and χ_2

are defined by:

$$\chi_1(\xi) = \begin{cases} 1 - \xi & 0 \leq \xi \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (24.47)$$

$$\chi_2(\xi) = \begin{cases} \xi & 0 \leq \xi \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



We may now extend the definition of ϕ_{e1} , ϕ_{e2} and $\hat{u}_e$ to the entire bar by simply using (24.45) for all x : $x_0 \leq x \leq x_N$. Note that the functions ϕ_{e1} and ϕ_{e2} satisfy, apart from (24.44),

$$\phi_{e1}(x) + \phi_{e2}(x) = \begin{cases} 1 & x_{e-1} \leq x \leq x_e \\ 0 & \text{otherwise} \end{cases} \quad (24.48)$$

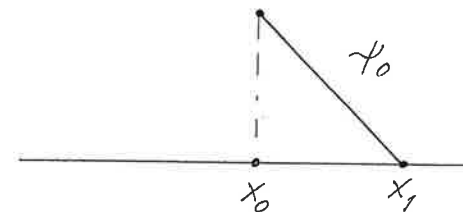
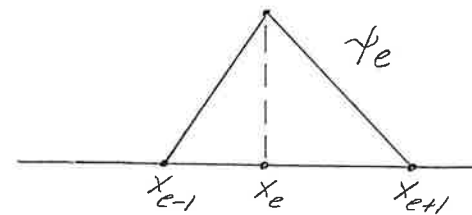
Also note that the functions ($2 \leq e \leq N-1$)

$$\psi_e(x) = \begin{cases} \phi_{e2}(x) & , & x_{e-1} \leq x \leq x_e \\ \phi_{(e+1)1}(x) & , & x_e \leq x \leq x_{e+1} \\ 0 & \text{otherwise} \end{cases} \quad (24.49)$$

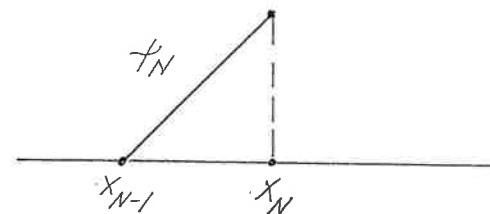
$$\psi_0(x) = \begin{cases} \phi_{11}(x) & , & x_0 \leq x \leq x_1 \\ 0 & , & \text{otherwise} \end{cases} \quad (24.50)$$

$$\psi_N(x) = \begin{cases} \phi_{N2}(x) & , & x_{N-1} \leq x \leq x_N \\ 0 & \text{otherwise} \end{cases} \quad (24.51)$$

are continuous.



"Hat - functions"



$$\sum_{e=0}^N \psi_e(x) = 1 \quad (\text{"partition of unity"})$$

We introduce the notation:

$$\bar{q}_e = \begin{pmatrix} u_{e-1} \\ u_e \end{pmatrix} \quad \bar{\Phi}_e = \begin{pmatrix} \phi_{e1} \\ \phi_{e2} \end{pmatrix} \text{ "shape vector" } \quad (24.52)$$

and then

$$\hat{u}_e = \bar{\Phi}_e^T \bar{q}_e \quad (24.53)$$

The displacement vector $\bar{q}$ is defined by: (u_i : node displacement)

$$\bar{q} = [u_0 \ u_1 \ \dots \ u_N]^T \quad (24.54)$$

We have the relation:

$$\bar{q}_e = \mathbb{L}_e \bar{q} \quad (24.55)$$

where $\mathbb{L}_e$ is the localization operator ($2 \times (N+1)$ -matrix):

$$\mathbb{L}_e = \begin{pmatrix} 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & & 0 & 1 & \dots & 0 \end{pmatrix} \quad (24.56)$$

↑
column e

We then have

$$\hat{u}_e = \bar{\Phi}_e^T \bar{q}_e = \bar{\Phi}_e^T \mathbb{L}_e \bar{q} \quad (24.57)$$

and

$$\hat{u}_N(x) = \sum_{e=1}^N \hat{u}_e(x) = \left(\sum_{e=1}^N \bar{\Phi}_e(x)^T \mathbb{L}_e \right) \bar{q} = \Phi(x)^T \bar{q} \quad (24.58)$$

where

$$\Phi = \sum_{e=1}^N \mathbb{L}_e^T \bar{\Phi}_e \quad (24.59)$$

Now, from (24.57) we have

$$\hat{u}_e' = \bar{\Phi}_e'^T \mathbb{L}_e \bar{q} \quad (24.60)$$

and then

$$\hat{u}_N'(x) = \Phi'(x)^T \bar{q} \quad (24.61)$$

With (24.58) and (24.61) inserted into the potential energy functional (24.11) we get:

$$V(\bar{q}) = \frac{EA}{2} \langle \bar{u}', \bar{u}' \rangle - \frac{\omega^2 m_0}{2} \langle \bar{u}, \bar{u} \rangle =$$

$$\frac{1}{2} \bar{q}^T EA \langle \Phi', \Phi'^T \rangle \bar{q} -$$

$$\omega^2 \frac{1}{2} \bar{q}^T m_0 \langle \Phi, \Phi^T \rangle \bar{q} = \frac{1}{2} \bar{q}^T \underline{K} \bar{q} -$$

$$\omega^2 \frac{1}{2} \bar{q}^T \underline{M} \bar{q} \quad (24.62)$$

where

$$\underline{M} = m_0 \langle \Phi, \Phi^T \rangle \quad (24.63)$$

is the mass-matrix and

$$\underline{K} = EA \langle \Phi', \Phi'^T \rangle \quad (24.64)$$

is the stiffness matrix of the finite element approximation.

Now from (24.59) and (24.63)

$$\underline{M} = \sum_{e=1}^N \mathbb{K}_e^T m_0 \langle \bar{\Phi}_e, \bar{\Phi}_e^T \rangle \mathbb{K}_e = \sum_{e=1}^N \mathbb{K}_e^T \underline{M}_e \mathbb{K}_e \quad (24.65)$$

where

$$\underline{M}_e = m_0 \langle \bar{\Phi}_e, \bar{\Phi}_e^T \rangle =$$

$$m_0 \int_0^L \begin{pmatrix} \phi_{e1}^2 & \phi_{e1} \phi_{e2} \\ \phi_{e2} \phi_{e1} & \phi_{e2}^2 \end{pmatrix} dx \quad (24.66)$$

is the element mass matrix and

$$\underline{K} = \sum_{e=1}^N \mathbb{K}_e^T EA \langle \bar{\Phi}_e', \bar{\Phi}_e'^T \rangle \mathbb{K}_e =$$

$$\sum_{e=1}^N \mathbb{K}_e^T \underline{K}_e \mathbb{K}_e \quad (24.67)$$

where

$$\underline{K}_e = EA \langle \bar{\Phi}_e', \bar{\Phi}_e'^T \rangle \quad (24.68)$$

is the element stiffness matrix.

The element mass-matrix elements are calculated according to:

$$m_{e11} = m_0 \int_{x_0}^{x_N} \phi_{e1}^2 dx = m_0 \int_{x_{e-1}}^{x_e} \left(\chi_1 \left(\frac{x - x_{e-1}}{h_e} \right) \right)^2 dx =$$

$$= m_0 l_e \int_0^1 \chi_1(\xi)^2 d\xi = m_0 l_e \int_0^1 (1-\xi)^2 d\xi =$$

$$\frac{m_0 l_e}{3}$$

$$m_{e12} = m_0 \int_{x_0}^{x_N} \phi_{e1} \phi_{e2} dx = m_0 l_e \int_0^1 \xi(1-\xi) d\xi =$$

$$\frac{m_0 l_e}{6} = m_{e21}$$

$$m_{ee2} = m_0 \int_{x_0}^{x_N} \phi_{e2}^2 dx = m_0 l_e \int_0^1 \xi^2 d\xi =$$

$$\frac{m_0 l_e}{3}$$

and then the element mass-matrix is

$$\underline{\underline{M}}_e = \frac{m_0 l_e}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad (24.69)$$

The element stiffness-matrix elements are calculated according to:

$$k_{e11} = EA \int_{x_0}^{x_N} (\phi'_{e1})^2 dx = EA \int_{x_{e-1}}^{x_e} \frac{1}{l_e^2} \left(\chi_1' \left(\frac{x-x_{e-1}}{l_e} \right) \right)^2 dx$$

$$= \frac{EA}{l_e} \int_0^1 (-1)^2 d\xi = \frac{EA}{l_e}$$

$$k_{e12} = EA \int_{x_0}^{x_N} \phi'_{e1} \phi'_{e2} dx = - \frac{EA}{l_e}$$

$$k_{ee2} = EA \int_{x_0}^{x_N} (\phi'_{e2})^2 dx = \frac{EA}{l_e}$$

and the element stiffness-matrix:

$$\underline{\underline{K}}_e = \frac{EA}{l_e} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad (24.70)$$

The structural mass-matrix:

$$\underline{\underline{M}} = \sum_{e=1}^N \underline{\underline{L}}_e^T \underline{\underline{M}}_e \underline{\underline{L}}_e \quad (24.71)$$

is an $(N+1) \times (N+1)$ -matrix. The algorithm defined by (24.71) is called the assembly operation. For the structural stiffness matrix we have:

$$\underline{K} = \sum_{e=1}^N \underline{K}_e^T \underline{K}_e \underline{K}_e \quad (24.72)$$

When all the finite elements have the same length:

$$\underline{K}_e = \underline{K} = \frac{x_N - x_0}{N} = \frac{L}{N} \quad (24.73)$$

We get for the "free-free bar":

$$\underline{M} = \frac{m_0 L}{6} \begin{pmatrix} 2 & 1 & 0 & & & \\ & 1 & 4 & 1 & & \\ & 0 & 1 & 4 & 1 & \\ & & & \ddots & \ddots & \\ & & & & 4 & 1 \\ & & & & 1 & 2 \end{pmatrix} \in \mathbb{R}^{(N+1) \times (N+1)} \quad (24.74)$$

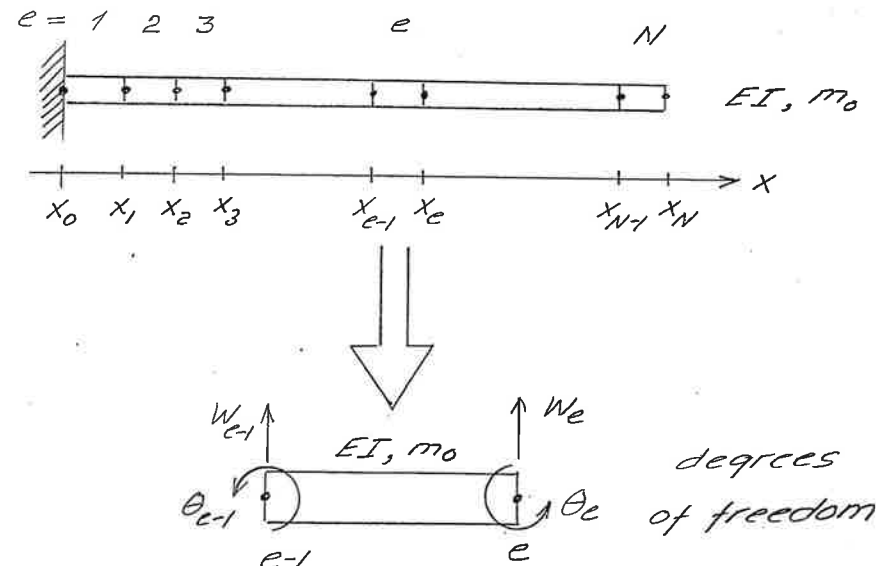
$$\underline{K} = \frac{EA}{L} \begin{pmatrix} 1 & -1 & 0 & & & \\ & -1 & 2 & -1 & & \\ & 0 & -1 & 2 & -1 & \\ & & & \ddots & \ddots & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(N+1) \times (N+1)}$$

Note that we also have, as an alternative to the representation (24.58),

$$\hat{u}_N(x) = \sum_{e=0}^N u_e \gamma_e(x) \quad (24.75)$$

where γ_e is defined in (24.49) - (24.51). From (24.75) the continuity of $\hat{u}_N$ is obvious.

We now look at the beam in bending:



The element no "e"

$$\{x \in \mathbb{R} : x_{e-1} \leq x \leq x_e\}, EI, m_0 \quad (24.76)$$

The nodes:

$$\| x_0, x_1, \dots, x_N \quad (24.77)$$

Nodal displacements:

$$\| w_0, w_1, \dots, w_N \quad (24.78)$$

Nodal rotations:

$$\| \theta_0, \theta_1, \dots, \theta_N \quad (24.79)$$

For the element displacement we assume ($l_e = x_e - x_{e-1}$):

$$\begin{aligned} \hat{w}_e(x) = & w_{e-1} \alpha_{e1} \left(\frac{x-x_{e-1}}{l_e} \right) + \theta_{e-1} \alpha_{e2} \left(\frac{x-x_{e-1}}{l_e} \right) + \\ & w_e \alpha_{e3} \left(\frac{x-x_{e-1}}{l_e} \right) + \theta_e \alpha_{e4} \left(\frac{x-x_{e-1}}{l_e} \right) \end{aligned} \quad (24.80)$$

where we choose the functions $\alpha_{e1}, \alpha_{e2}, \alpha_{e3}, \alpha_{e4}$ so that:

$$\begin{aligned} \hat{w}_e(x_{e-1}) &= w_{e-1}, \quad \hat{w}_e(x_e) = w_e \\ \hat{w}'_e(x_{e-1}) &= \theta_{e-1}, \quad \hat{w}'_e(x_e) = \theta_e \end{aligned} \quad (24.81)$$

or equivalently:

$$\alpha_{e1}(0) = 1, \quad \alpha'_{e1}(0) = 0 \quad (24.82)$$

$$\alpha_{e1}(1) = 0, \quad \alpha'_{e1}(1) = 0$$

$$\alpha_{e2}(0) = 0, \quad \alpha'_{e2}(0) = l_e \quad (24.83)$$

$$\alpha_{e2}(1) = 0, \quad \alpha'_{e2}(1) = 0$$

$$\alpha_{e3}(0) = 0, \quad \alpha'_{e3}(0) = 0 \quad (24.84)$$

$$\alpha_{e3}(1) = 1, \quad \alpha'_{e3}(1) = 0$$

$$\alpha_{e4}(0) = 0, \quad \alpha'_{e4}(0) = 0 \quad (24.85)$$

$$\alpha_{e4}(1) = 0, \quad \alpha'_{e4}(1) = l_e$$

If we for instance assume that:

$$\alpha_{e2}(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3 \quad (24.86)$$

the conditions (24.83) imply

$$a_0 = 0$$

$$a_1 = l_e$$

$$a_0 + a_1 + a_2 + a_3 = 0$$

$$a_1 + 2a_2 + 3a_3 = 0$$

$$\Rightarrow a_2 = -2a_3, \quad a_3 = a_e$$

and then

$$\alpha_{e2}(\xi) = a_e \xi - 2a_e \xi^2 + a_e \xi^3 = a_e \xi(1-\xi)^2 \quad (24.87)$$

In the same way we find that:

$$\alpha_{e1}(\xi) = 1 - 3\xi^2 + 2\xi^3$$

$$\alpha_{e3}(\xi) = \xi^2(3-2\xi)$$

$$\alpha_{e4}(\xi) = a_e \xi^2(\xi-1)$$

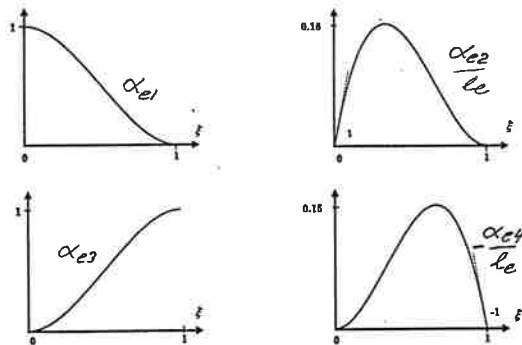


Figure 5.9.11
Shape functions:
Hermitian polynomials of the third order

We use the notation:

$$\bar{\alpha}_e = [\alpha_{e1} \alpha_{e2} \alpha_{e3} \alpha_{e4}]^T \quad \text{shape function vector}$$

$$\bar{q}_e = [w_{e-1} \theta_{e-1} w_e \theta_e]^T \quad \text{element displ. vector} \quad (24.89)$$

and then

$$\hat{w}_e = \bar{\alpha}_e^T \bar{q}_e \quad (24.90)$$

Now with the structural displacement vector defined by: ($2(N+1)$ DOF)

$$\bar{q} = [w_0 \theta_0 w_1 \theta_1, \dots, w_{N-1} \theta_{N-1} w_N \theta_N]^T \quad (24.91)$$

we have the relation

$$\bar{q}_e = \mathbb{K}_e \bar{q} \quad (24.92)$$

where $\mathbb{K}_e$ is the localization operator ($4 \times 2(N+1)$ - matrix):

$$\mathbb{K}_e = \begin{pmatrix} 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \end{pmatrix} \quad (24.93)$$

$\downarrow$ column $2e+2$
 $\uparrow$ column $2e-1$

and then

$$\hat{W}_e = \bar{\alpha}_e^T \mathcal{L}_e \bar{q}$$

$$\hat{W} = \left(\sum_{e=1}^N \bar{\alpha}_e^T \mathcal{L}_e \right) \bar{q} = \bar{\Phi}^T \bar{q}$$

$$\bar{\Phi} = \sum_{e=1}^N \mathcal{L}_e^T \bar{\alpha}_e \quad (24.94)$$

$$\hat{W}' = \bar{\Phi}'^T \bar{q}$$

$$\hat{W}'' = \bar{\Phi}''^T \bar{q}$$

The potential energy functional:

$$V(\hat{W}) = \frac{1}{2} EI \langle \hat{W}'', \hat{W}'' \rangle - \frac{1}{2} \omega^2 m_0 \langle \hat{W}, \hat{W} \rangle =$$

$$\frac{1}{2} \bar{q}^T EI \langle \bar{\Phi}'', \bar{\Phi}'' \rangle \bar{q} - \omega^2 \frac{1}{2} \bar{q}^T \langle \bar{\Phi}, \bar{\Phi} \rangle \bar{q} =$$

$$\frac{1}{2} \bar{q}^T \underline{K} \bar{q} - \omega^2 \frac{1}{2} \bar{q}^T \underline{M} \bar{q} \quad (24.95)$$

where (the assembly)

$$\underline{M} = m_0 \langle \bar{\Phi}, \bar{\Phi} \rangle = \sum_{e=1}^N \mathcal{L}_e^T \underline{M}_e \mathcal{L}_e \quad (24.96)$$

$$\underline{M}_e = m_0 \langle \bar{\alpha}_e, \bar{\alpha}_e^T \rangle$$

$$\underline{K} = EI \langle \bar{\Phi}'', \bar{\Phi}'' \rangle = \sum_{e=1}^N \mathcal{L}_e^T \underline{K}_e \mathcal{L}_e \quad (24.97)$$

$$\underline{K}_e = EI \langle \bar{\alpha}_e'', \bar{\alpha}_e''^T \rangle$$

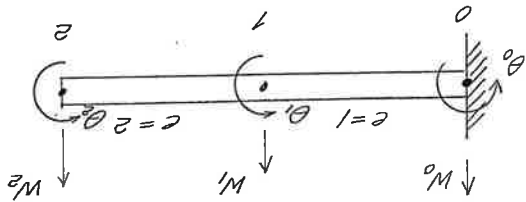
A calculation gives:

$$\underline{M}_e = \frac{m_0 l_e}{420} \begin{pmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{pmatrix}$$

(24.98)

$$\underline{K}_e = \frac{EI}{l_e^3} \begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}$$

EX 18 The clamped-free beam



$$L_c = \frac{L}{2}$$

From (24.98) we get ($L_c = \frac{L}{2}$)

$$\bar{M}_1 = \frac{m_0 L}{840} \begin{pmatrix} 156 & 11L & 54 & -\frac{13}{3}L \\ 11L & L^2 & \frac{13}{3}L & -\frac{4}{3}L^2 \\ 54 & \frac{13}{3}L & 156 & -11L \\ -\frac{13}{3}L & -\frac{4}{3}L^2 & -11L & L^2 \end{pmatrix} \begin{matrix} W_0 \\ \theta_0 \\ W_1 \\ \theta_1 \end{matrix}$$

$$\bar{K}_1 = \frac{8EI}{L^3} \begin{pmatrix} 12 & 3L & -12 & 3L \\ 3L & L^2 & -3L & \frac{7}{2}L^2 \\ -12 & -3L & 12 & -3L \\ 3L & \frac{7}{2}L^2 & -3L & L^2 \end{pmatrix} \begin{matrix} W_0 \\ \theta_0 \\ W_1 \\ \theta_1 \end{matrix}$$

Boundary conditions require that:

$$W_0 = 0, \theta_0 = 0$$

Cancellation of the rows and columns associated with these two coordinates

yields:

$$\bar{M}_1 = \frac{m_0 L}{840} \begin{pmatrix} 156 & -11L \\ -11L & L^2 \end{pmatrix} \begin{matrix} W_1 \\ \theta_1 \end{matrix}$$

$$\bar{K}_1 = \frac{8EI}{L^3} \begin{pmatrix} 12 & -3L \\ -3L & L^2 \end{pmatrix} \begin{matrix} W_1 \\ \theta_1 \end{matrix}$$

For the second element we get

$$\bar{M}_2 = \frac{m_0 L}{840} \begin{pmatrix} 156 & 11L & 54 & -\frac{13}{3}L \\ 11L & L^2 & \frac{13}{3}L & -\frac{4}{3}L^2 \\ 54 & \frac{13}{3}L & 156 & -11L \\ -\frac{13}{3}L & -\frac{4}{3}L^2 & -11L & L^2 \end{pmatrix} \begin{matrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{matrix}$$

$$\bar{K}_2 = \frac{8EI}{L^3} \begin{pmatrix} 12 & 3L & -12 & 3L \\ 3L & L^2 & -3L & \frac{7}{2}L^2 \\ -12 & -3L & 12 & -3L \\ 3L & \frac{7}{2}L^2 & -3L & L^2 \end{pmatrix} \begin{matrix} W_1 \\ \theta_1 \\ W_2 \\ \theta_2 \end{matrix}$$

Assembly of element mass matrices gives structural mass matrix:

$$\bar{M} = \frac{m_0 L}{840} \begin{pmatrix} 312 & 0 & 54 & -6.5L \\ 0 & 2L^2 & 6.5L & -0.35L^2 \\ 54 & 6.5L & 1.56 & -71L \\ -6.5L & -0.35L^2 & -11L & L^2 \end{pmatrix} \begin{matrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{matrix}$$

Assembly of element stiffness matrices gives the structural stiffness matrix:

$$\bar{K} = \frac{EI}{L^3} \begin{pmatrix} 24 & 0 & -12 & 3L \\ 0 & 2L^2 & -3L & \frac{7}{2}L^2 \\ -12 & -3L & 12 & -3L \\ 3L & \frac{7}{2}L^2 & -3L & L^2 \end{pmatrix} \begin{matrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{matrix}$$

APPENDICES

Appendix A.1: Derivatives of vectors and matrices

We use the following notation. If we have a real-valued function

$$f = f(q_1, q_2, \dots, q_n) \in \mathbb{R} \quad (\text{A.1.1})$$

then

$$\frac{\partial f}{\partial q} = \left[\frac{\partial f}{\partial q_1} \quad \frac{\partial f}{\partial q_2} \quad \dots \quad \frac{\partial f}{\partial q_n} \right] \in \mathbb{R}^{1 \times n} \quad (\text{A.1.2})$$

that is the i^{th} component of this vector is given by

$$\left(\frac{\partial f}{\partial q} \right)_i = \frac{\partial f}{\partial q_i}, \quad i = 1, \dots, n \quad (\text{A.1.3})$$

If we have the vector-valued function

$$a = a(q) = a(q_1, q_2, \dots, q_n) = [a_1 \quad a_2 \quad \dots \quad a_n]^T \in \mathbb{R}^{n \times 1} \quad (\text{A.1.4})$$

where

$$a_i = a_i(q) = a_i(q_1, q_2, \dots, q_n), \quad i = 1, \dots, n \quad (\text{A.1.5})$$

then the so-called *Jacobian* is given by

$$\frac{\partial a}{\partial q} = \begin{bmatrix} \frac{\partial a_1}{\partial q_1} & \frac{\partial a_1}{\partial q_2} & \dots & \frac{\partial a_1}{\partial q_n} \\ \frac{\partial a_2}{\partial q_1} & \frac{\partial a_2}{\partial q_2} & \dots & \frac{\partial a_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_n}{\partial q_1} & \frac{\partial a_n}{\partial q_2} & \dots & \frac{\partial a_n}{\partial q_n} \end{bmatrix} \in \mathbb{R}^{n \times n} \quad (\text{A.1.6})$$

That is

$$\left(\frac{\partial a}{\partial q} \right)_{ij} = \frac{\partial a_i}{\partial q_j} \quad (\text{A.1.7})$$

Note that

$$\frac{\partial q}{\partial q} = \begin{bmatrix} \frac{\partial q_1}{\partial q_1} & \frac{\partial q_1}{\partial q_2} & \dots & \frac{\partial q_1}{\partial q_n} \\ \frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial q_2} & \dots & \frac{\partial q_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_n}{\partial q_1} & \frac{\partial q_n}{\partial q_2} & \dots & \frac{\partial q_n}{\partial q_n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underline{1} \quad (\text{A.1.8})$$

i.e. the unit matrix. This may be expressed as

$$\left(\frac{\partial q}{\partial q}\right)_{ij} = \delta_{ij} \quad (\text{A.1.9})$$

With $u = [u_1 \ u_2 \ \dots \ u_n]^T \in \mathbb{R}^{n \times 1}$ and $v = [v_1 \ v_2 \ \dots \ v_n]^T \in \mathbb{R}^{n \times 1}$ we have

$$u^T \frac{\partial a}{\partial q} v = \sum_{i,j=1}^n \frac{\partial a_i}{\partial q_j} u_i v_j \quad (\text{A.1.10})$$

If we have the matrix-valued function

$$A = A(q) = A(q_1, q_2, \dots, q_n) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \in \mathbb{R}^{n \times n} \quad (\text{A.1.11})$$

where

$$A_{ij} = A_{ij}(q) = A_{ij}(q_1, q_2, \dots, q_n) \quad (\text{A.1.12})$$

then

$$u^T \left(\frac{\partial A}{\partial q} v\right) w = \sum_{i,j,k=1}^n \frac{\partial A_{ij}}{\partial q_k} u_i v_j w_k \quad (\text{A.1.13})$$

or

$$\left(\frac{\partial A}{\partial q} v\right)_{ik} = \sum_{j=1}^n \frac{\partial A_{ij}}{\partial q_k} v_j \quad (\text{A.1.14})$$

Note that if A is a symmetric matrix, i.e. $A^T = A$, then

$$u^T \left(\frac{\partial A}{\partial q} v\right) w = \sum_{i,j,k=1}^n \frac{\partial A_{ij}}{\partial q_k} u_i v_j w_k = \sum_{i,j,k=1}^n \frac{\partial A_{ji}}{\partial q_k} v_j u_i w_k = v^T \left(\frac{\partial A}{\partial q} u\right) w \quad (\text{A.1.15})$$

We have the following *derivation rules*:

Proposition A.1.1 With $a = [a_1 \ a_2 \ \dots \ a_n]^T$, $u = [u_1 \ u_2 \ \dots \ u_n]^T$ and A an $n \times n$ -matrix, according to (A.1.11), all depending on q . Then

$$\frac{\partial}{\partial q}(a^T u) = u^T \frac{\partial a}{\partial q} + a^T \frac{\partial u}{\partial q} \quad (\text{A.1.16})$$

$$\frac{\partial}{\partial q}(Au) = \frac{\partial A}{\partial q}u + A \frac{\partial u}{\partial q} \quad (\text{A.1.17})$$

$$\frac{\partial}{\partial q}(u^T Av) = u^T \frac{\partial A}{\partial q}v + u^T A \frac{\partial v}{\partial q} + v^T A^T \frac{\partial u}{\partial q} \quad (\text{A.1.18})$$

Proof: A straightforward calculation gives

$$\begin{aligned} \frac{\partial}{\partial q}(a^T u) &= \frac{\partial}{\partial q} \left(\sum_{i=1}^n a_i u_i \right) = \left[\frac{\partial}{\partial q_1} \left(\sum_{i=1}^n a_i u_i \right) \ \frac{\partial}{\partial q_2} \left(\sum_{i=1}^n a_i u_i \right) \ \dots \ \frac{\partial}{\partial q_n} \left(\sum_{i=1}^n a_i u_i \right) \right] = \\ &= \left[\left(\sum_{i=1}^n \frac{\partial a_i}{\partial q_1} u_i + a_i \frac{\partial u_i}{\partial q_1} \right) \left(\sum_{i=1}^n \frac{\partial a_i}{\partial q_2} u_i + a_i \frac{\partial u_i}{\partial q_2} \right) \ \dots \ \left(\sum_{i=1}^n \frac{\partial a_i}{\partial q_n} u_i + a_i \frac{\partial u_i}{\partial q_n} \right) \right] = \\ &= \left[\sum_{i=1}^n \frac{\partial a_i}{\partial q_1} u_i \ \sum_{i=1}^n \frac{\partial a_i}{\partial q_2} u_i \ \dots \ \sum_{i=1}^n \frac{\partial a_i}{\partial q_n} u_i \right] + \left[\sum_{i=1}^n a_i \frac{\partial u_i}{\partial q_1} \ \sum_{i=1}^n a_i \frac{\partial u_i}{\partial q_2} \ \dots \ \sum_{i=1}^n a_i \frac{\partial u_i}{\partial q_n} \right] = \\ &= [u_1 \ u_2 \ \dots \ u_n] \begin{bmatrix} \frac{\partial a_1}{\partial q_1} & \frac{\partial a_1}{\partial q_2} & \dots & \frac{\partial a_1}{\partial q_n} \\ \frac{\partial a_2}{\partial q_1} & \frac{\partial a_2}{\partial q_2} & \dots & \frac{\partial a_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_n}{\partial q_1} & \frac{\partial a_n}{\partial q_2} & \dots & \frac{\partial a_n}{\partial q_n} \end{bmatrix} + [a_1 \ a_2 \ \dots \ a_n] \begin{bmatrix} \frac{\partial u_1}{\partial q_1} & \frac{\partial u_1}{\partial q_2} & \dots & \frac{\partial u_1}{\partial q_n} \\ \frac{\partial u_2}{\partial q_1} & \frac{\partial u_2}{\partial q_2} & \dots & \frac{\partial u_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial u_n}{\partial q_1} & \frac{\partial u_n}{\partial q_2} & \dots & \frac{\partial u_n}{\partial q_n} \end{bmatrix} = \\ &= u^T \frac{\partial a}{\partial q} + a^T \frac{\partial u}{\partial q} \quad (\text{A.1.19}) \end{aligned}$$

and this proves (A.1.16). Furthermore

$$\frac{\partial}{\partial q}(Au) = \frac{\partial}{\partial q} \left(\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \right) =$$

$$\begin{aligned}
 &= \frac{\partial}{\partial q} \begin{bmatrix} \sum_{i=1}^n A_{1j} u_j \\ \sum_{j=1}^n A_{2j} u_j \\ \vdots \\ \sum_{j=1}^n A_{nj} u_j \end{bmatrix} = \begin{bmatrix} \frac{\partial(\sum_{i=1}^n A_{1j} u_j)}{\partial q_1} & \frac{\partial(\sum_{i=1}^n A_{1j} u_j)}{\partial q_2} & \dots & \frac{\partial(\sum_{i=1}^n A_{1j} u_j)}{\partial q_n} \\ \frac{\partial(\sum_{j=1}^n A_{2j} u_j)}{\partial q_1} & \frac{\partial(\sum_{j=1}^n A_{2j} u_j)}{\partial q_2} & \dots & \frac{\partial(\sum_{j=1}^n A_{2j} u_j)}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(\sum_{j=1}^n A_{nj} u_j)}{\partial q_1} & \frac{\partial(\sum_{j=1}^n A_{nj} u_j)}{\partial q_2} & \dots & \frac{\partial(\sum_{j=1}^n A_{nj} u_j)}{\partial q_n} \end{bmatrix} = \\
 &= \begin{bmatrix} \sum_{j=1}^n \frac{\partial A_{1j}}{\partial q_1} u_j + A_{1j} \frac{\partial u_j}{\partial q_1} & \sum_{j=1}^n \frac{\partial A_{1j}}{\partial q_2} u_j + A_{1j} \frac{\partial u_j}{\partial q_2} & \dots & \sum_{j=1}^n \frac{\partial A_{1j}}{\partial q_n} u_j + A_{1j} \frac{\partial u_j}{\partial q_n} \\ \sum_{j=1}^n \frac{\partial A_{2j}}{\partial q_1} u_j + A_{2j} \frac{\partial u_j}{\partial q_1} & \sum_{j=1}^n \frac{\partial A_{2j}}{\partial q_2} u_j + A_{2j} \frac{\partial u_j}{\partial q_2} & \dots & \sum_{j=1}^n \frac{\partial A_{2j}}{\partial q_n} u_j + A_{2j} \frac{\partial u_j}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n \frac{\partial A_{nj}}{\partial q_1} u_j + A_{nj} \frac{\partial u_j}{\partial q_1} & \sum_{j=1}^n \frac{\partial A_{nj}}{\partial q_2} u_j + A_{nj} \frac{\partial u_j}{\partial q_2} & \dots & \sum_{j=1}^n \frac{\partial A_{nj}}{\partial q_n} u_j + A_{nj} \frac{\partial u_j}{\partial q_n} \end{bmatrix} = \\
 &= \begin{bmatrix} \sum_{j=1}^n \frac{\partial A_{1j}}{\partial q_1} u_j & \sum_{j=1}^n \frac{\partial A_{1j}}{\partial q_2} u_j & \dots & \sum_{j=1}^n \frac{\partial A_{1j}}{\partial q_n} u_j \\ \sum_{j=1}^n \frac{\partial A_{2j}}{\partial q_1} u_j & \sum_{j=1}^n \frac{\partial A_{2j}}{\partial q_2} u_j & \dots & \sum_{j=1}^n \frac{\partial A_{2j}}{\partial q_n} u_j \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n \frac{\partial A_{nj}}{\partial q_1} u_j & \sum_{j=1}^n \frac{\partial A_{nj}}{\partial q_2} u_j & \dots & \sum_{j=1}^n \frac{\partial A_{nj}}{\partial q_n} u_j \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^n A_{1j} \frac{\partial u_j}{\partial q_1} & \sum_{j=1}^n A_{1j} \frac{\partial u_j}{\partial q_2} & \dots & \sum_{j=1}^n A_{1j} \frac{\partial u_j}{\partial q_n} \\ \sum_{j=1}^n A_{2j} \frac{\partial u_j}{\partial q_1} & \sum_{j=1}^n A_{2j} \frac{\partial u_j}{\partial q_2} & \dots & \sum_{j=1}^n A_{2j} \frac{\partial u_j}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n A_{nj} \frac{\partial u_j}{\partial q_1} & \sum_{j=1}^n A_{nj} \frac{\partial u_j}{\partial q_2} & \dots & \sum_{j=1}^n A_{nj} \frac{\partial u_j}{\partial q_n} \end{bmatrix} = \\
 &= \frac{\partial A}{\partial q} u + A \frac{\partial u}{\partial q} \tag{A.1.20}
 \end{aligned}$$

and this proves (A.1.17). Using (A.1.16) and (A.1.17) it follows that

$$\begin{aligned}
 \frac{\partial}{\partial q} (u^T A v) &= u^T \frac{\partial A v}{\partial q} + (A v)^T \frac{\partial u}{\partial q} = u^T \left(\frac{\partial A}{\partial q} v + A \frac{\partial v}{\partial q} \right) + v^T A^T \frac{\partial u}{\partial q} = \\
 &= u^T \frac{\partial A}{\partial q} v + u^T A \frac{\partial v}{\partial q} + v^T A^T \frac{\partial u}{\partial q} \tag{A.1.21}
 \end{aligned}$$

and this proves (A.1.18). $\square$

Corollary A.1.1

$$\frac{\partial}{\partial q}(q^T A q) = q^T \frac{\partial A}{\partial q} q + q^T (A + A^T) \quad (\text{A.1.22})$$

and if

$$\frac{\partial A}{\partial q} = 0 \text{ and } A^T = A \quad (\text{A.1.23})$$

then

$$\frac{\partial}{\partial q}(q^T A q) = 2q^T A \quad (\text{A.1.24})$$

Proof: Taking $u = v = q$ in (A.1.18) gives

$$\frac{\partial}{\partial q}(q^T A q) = q^T \frac{\partial A}{\partial q} q + q^T A \frac{\partial q}{\partial q} + q^T A^T \frac{\partial q}{\partial q} = q^T \frac{\partial A}{\partial q} q + q^T (A + A^T) \quad (\text{A.1.25})$$

where we have used (A.1.8). This proves (A.1.22). If

$$\frac{\partial A}{\partial q} = 0 \text{ and } A^T = A \quad (\text{A.1.26})$$

then, from (A.1.22) it follows that

$$\frac{\partial}{\partial q}(q^T A q) = 2q^T A \quad (\text{A.1.27})$$

and this proves (A.1.24). $\square$

Theorem A.1.1 The real-valued function

$$T = T(q, \dot{q}) = \frac{1}{2}(f(q) + a(q)^T \dot{q} + \dot{q}^T A(q) \dot{q}) \quad (\text{A.1.28})$$

has the following derivatives

$$\frac{\partial T}{\partial q} = \frac{1}{2} \left(\frac{\partial f}{\partial q} + \dot{q}^T \frac{\partial a}{\partial q} + \dot{q}^T \frac{\partial A^T}{\partial q} \dot{q} \right) \quad (\text{A.1.29})$$

$$\frac{\partial T}{\partial \dot{q}} = \frac{1}{2} (a^T + \dot{q}^T (A(q) + A^T(q)))$$

Proof: Applying the derivation rules in proposition A.1.1 and Corollary A.1.1 one finds that

$$\frac{\partial T}{\partial q} = \frac{1}{2} \left(\frac{\partial f}{\partial q} + \dot{q}^T \frac{\partial a}{\partial q} + \dot{q} \frac{\partial(A\dot{q})}{\partial q} \right) = \frac{1}{2} \left(\frac{\partial f}{\partial q} + \dot{q}^T \frac{\partial a}{\partial q} + \dot{q}^T \frac{\partial A}{\partial q} \dot{q} \right) \quad (\text{A.1.30})$$

and

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}} &= \frac{1}{2} \left(\frac{\partial f}{\partial \dot{q}} + \dot{q}^T \frac{\partial a}{\partial \dot{q}} + a^T \frac{\partial \dot{q}}{\partial \dot{q}} + (A(q)\dot{q})^T \frac{\partial \dot{q}}{\partial \dot{q}} + \dot{q}^T \frac{\partial A(q)\dot{q}}{\partial \dot{q}} \right) = \\ &= \frac{1}{2} (a^T + (A(q)\dot{q})^T + \dot{q}^T A(q)) = \frac{1}{2} (a^T + \dot{q}^T (A(q) + A^T(q))) \end{aligned} \quad (\text{A.1.31})$$

where we have used

$$\frac{\partial f}{\partial \dot{q}} = 0 \quad \text{and} \quad \left(\frac{\partial \dot{q}}{\partial \dot{q}} \right)_{ij} = \delta_{ij} \quad (\text{A.1.32})$$

and this proves the Theorem. $\square$

Appendix A.2: Lagrange's equations and their linearization

This text may be seen as a complement to the pages 71 -76 in the Lecture Notes. It contains more details on the linearization procedure.

Lagrange's equations may be written

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = Q_i, \quad i = 1, \dots, n \quad (\text{A.2.1})$$

where

$$T = T(t, q, \dot{q}) = T_0 + T_1 + T_2 \quad (\text{A.2.2})$$

is the *kinetic energy* and

$$Q_i = \int_{\mathcal{B}} \mathbf{f} \cdot \frac{\partial \mathbf{r}}{\partial q_i} dm, \quad i = 1, \dots, n \quad (\text{A.2.3})$$

is the *generalized force*. We now divide the generalized forces into two parts, the *conservative* and the *non-conservative*,

$$Q_i = Q_i^{con} + Q_i^{non} \quad (\text{A.2.4})$$

and assume that

$$Q_i^{con} = -\frac{\partial V}{\partial q^k} \quad (\text{A.2.5})$$

where

$$V = V(t, q) \quad (\text{A.2.6})$$

is the potential energy containing all conservative forces, such as the *gravity and elastic forces*. Introducing the Lagrangian

$$L = L(t, q, \dot{q}) = T - V \quad (\text{A.2.7})$$

Lagrange's equations may be written

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i^{non}, \quad i = 1, \dots, n \quad (\text{A.2.8})$$

We may split the generalized non-conservative force Q_i^{non} according to

$$Q_i^{non} = Q_i^{int} + Q_i^{ext} \quad (\text{A.2.9})$$

i.e. into a sum of internal and external forces. It may be assumed that

$$Q_i^{int} = -\frac{\partial D}{\partial \dot{q}_i} \quad (\text{A.2.10})$$

where

$$D = D(t, q, \dot{q}) \quad (\text{A.2.11})$$

is the so-called *dissipation function*. Lagrange's equations may then be written

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) - \frac{\partial L}{\partial q^i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i^{ext}, \quad i = 1, \dots, n \quad (\text{A.2.12})$$

Using the *relative Lagrangian*

$$L = T_2 - V = \frac{1}{2} \sum_{i,j=1}^n a_{ij} \dot{q}_i \dot{q}_j - V \quad (\text{A.2.13})$$

then Lagrange's equations take the form

$$\frac{d}{dt} \left(\frac{\partial L_2}{\partial \dot{q}^i} \right) - \frac{\partial L_2}{\partial q^i} + \frac{\partial D}{\partial \dot{q}^i} = -K_i^t - K_i^g + Q_i^{ext}, \quad i = 1, \dots, n \quad (\text{A.2.14})$$

where the *transport inertia force* K_i^t and the *gyroscopic inertia force* K_i^g are defined by

$$K_i^t = \frac{\partial}{\partial t} \left(\frac{\partial T_1}{\partial \dot{q}_i} \right) - \frac{\partial T_0}{\partial q_i} = \frac{\partial a_{0i}}{\partial t} - \frac{1}{2} \frac{\partial a_{00}}{\partial q_i} \quad \text{and} \quad K_i^g = \sum_{j=1}^n \frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} \dot{q}_j - \frac{\partial T_1}{\partial q_i} = \sum_{j=1}^n \tilde{g}_{ij} \dot{q}_j \quad (\text{A.2.15})$$

respectively, and where

$$\tilde{g}_{ij} = \frac{\partial^2 T_1}{\partial q_j \partial \dot{q}_i} - \frac{\partial^2 T_1}{\partial q_i \partial \dot{q}_j} = \frac{\partial a_{0i}}{\partial q_j} - \frac{\partial a_{0j}}{\partial q_i} \quad (\text{A.2.16})$$

Note that the matrix $(\tilde{g}_{ij})$ is *anti-symmetric*

$$(\tilde{g}_{ij})^T = -(\tilde{g}_{ij}) \quad (\text{A.2.17})$$

i.e. $\tilde{g}_{ij} = -\tilde{g}_{ji}$. A straightforward calculation gives the identity

$$\frac{d}{dt} \left(\frac{\partial L_2}{\partial \dot{q}_i} \right) - \frac{\partial L_2}{\partial q_i} = \sum_{j=1}^n a_{ij} \ddot{q}_j + \sum_{j,k=1}^n C_{ijk} \dot{q}_j \dot{q}_k + \sum_{j=1}^n \frac{\partial a_{ij}}{\partial t} \dot{q}_j + \frac{\partial V}{\partial q_i} \quad (\text{A.2.18})$$

where

$$C_{ijk} = \frac{1}{2} \left(\frac{\partial a_{ij}}{\partial q^k} - \frac{\partial a_{jk}}{\partial q^i} + \frac{\partial a_{ki}}{\partial q^j} \right), \quad i, j, k = 1, \dots, n \quad (\text{A.2.19})$$

and then Lagrange's equations read

$$\sum_{j=1}^n a_{ij} \ddot{q}_j + \sum_{j,k=1}^n C_{ijk} \dot{q}_j \dot{q}_k + \sum_{j=1}^n \frac{\partial a_{ij}}{\partial t} \dot{q}_j + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = -K_i^t - K_i^g + Q_i^{ext}, \quad i = 1, \dots, n \quad (\text{A.2.20})$$

This is a set of n second order ordinary differential equations for the determination of *the motion*

$$q_i = q_i(t), \quad i = 1, \dots, n \quad (\text{A.2.21})$$

once the external force

$$Q_i^{ext} = Q_i^{ext}(t) \quad (\text{A.2.22})$$

and initial conditions

$$q_i(0) = q_i^0, \quad \dot{q}_i(0) = \dot{q}_i^0 \quad (\text{A.2.23})$$

are given. Here $q_i^0, \dot{q}_i^0$ $i = 1, \dots, n$ are given constants. The mechanical system is called *non-disturbed* if there is no external force, i.e. if

$$Q_i^{ext}(t) = 0, \quad t \geq 0 \quad (\text{A.2.24})$$

Lagrange's equations then read

$$\sum_{j=1}^n a_{ij} \ddot{q}_j + \sum_{j,k=1}^n C_{ijk} \dot{q}_i \dot{q}_j + \sum_{j=1}^n \frac{\partial a_{ij}}{\partial t} \dot{q}_j + \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}^i} = -K_i^t - K_i^g, \quad i = 1, \dots, n \quad (\text{A.2.25})$$

and this represents *the free motion* of the mechanical system.

A fixed configuration $q = q^0$ is an *equilibrium configuration* to the non-disturbed system if

$$q_i(t) = q_i^0, \quad t \geq 0 \quad (\text{A.2.26})$$

is a solution to (A.2.25). If we insert $q_i(t) = q_i^0$, $(\dot{q}_i(t) = 0, \ddot{q}_i(t) = 0)$ into (A.2.25) then we obtain the equation

$$\frac{\partial V}{\partial q_i}(t, q^0, 0) = -K_i^t(t, q^0, 0) \quad (\text{A.2.27})$$

where we have assume that

$$\frac{\partial D}{\partial \dot{q}_i}(t, q^0, 0) = 0 \quad (\text{A.2.28})$$

and used the fact that

$$K_i^g(t, q^0, 0) = 0 \quad (\text{A.2.29})$$

The equilibrium equation (A.2.27) is equivalent to

$$\frac{\partial V^*}{\partial q_i} = -\frac{\partial}{\partial t} \left(\frac{\partial T_i}{\partial \dot{q}_i} \right) = -\frac{\partial a_{0i}}{\partial t} \quad (\text{A.2.30})$$

where

$$V^* = V - T_0 \quad (\text{A.2.31})$$

is the so-called *modified potential*. Now if

$$\frac{\partial a_{0i}}{\partial t}(t, q^0) = 0 \quad (\text{A.2.32})$$

which obviously is true if the coordinate system is scleronomous, then the equilibrium equation reads

$$\frac{\partial V^*}{\partial q_i}(t, q^0) = 0 \quad (\text{A.2.33})$$

which means that the equilibrium configurations of the undisturbed system are the configurations for which the modified potential is *stationary*.

We take Lagrange's equations on the format

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = Q_i^{int} + Q_i^{ext}, \quad i = 1, \dots, n \quad (\text{A.2.34})$$

and assume that we have an *equilibrium state*: $q_i = q_i^0$, $\dot{q}_i = 0$. We introduce the matrices

$$\bar{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}, \quad \dot{\bar{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{pmatrix}, \quad \ddot{\bar{q}} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{pmatrix}, \quad \bar{q}^0 = \begin{pmatrix} q_1^0 \\ q_2^0 \\ \vdots \\ q_n^0 \end{pmatrix} \quad (\text{A.2.35})$$

an the *deviation from the equilibrium configuration*

$$\bar{r} = \bar{q} - \bar{q}^0 = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \quad (\text{A.2.36})$$

A Taylor expansion of the Lagrangian gives

$$\begin{aligned} L(t, \bar{q}, \dot{\bar{q}}) &= L(t, \bar{q}^0 + \bar{r}, \dot{\bar{q}}^0 + \dot{\bar{r}}) = \\ &= L(t, \bar{q}^0, \dot{\bar{q}}^0) + \left(\frac{\partial L}{\partial \bar{q}}\right)_0 \bar{r} + \left(\frac{\partial L}{\partial \dot{\bar{q}}}\right)_0 \dot{\bar{r}} + \frac{1}{2} \bar{r}^T \left(\frac{\partial^2 L}{\partial \bar{q}^2}\right)_0 \bar{r} + \bar{r}^T \left(\frac{\partial^2 L}{\partial \bar{q} \partial \dot{\bar{q}}}\right)_0 \dot{\bar{r}} + \frac{1}{2} \dot{\bar{r}}^T \left(\frac{\partial^2 L}{\partial \dot{\bar{q}}^2}\right)_0 \dot{\bar{r}} + \dots \end{aligned} \quad (\text{A.2.37})$$

where higher order terms in $\bar{r}$ and $\dot{\bar{r}}$ have been neglected, We use a notation where, for instance,

$$\left(\frac{\partial L}{\partial \bar{q}}\right)_0 = \left(\frac{\partial L}{\partial q_1} \quad \frac{\partial L}{\partial q_2} \quad \dots \quad \frac{\partial L}{\partial q_n} \right)_0 = \frac{\partial L}{\partial \bar{q}}(t, \bar{q}^0, \dot{\bar{q}}^0) \quad (\text{A.2.38})$$

and

$$\left(\frac{\partial^2 L}{\partial \bar{q}^2}\right)_0 = \begin{pmatrix} \frac{\partial^2 L}{\partial q_1^2} & \frac{\partial^2 L}{\partial q_1 \partial q_2} & \cdots & \frac{\partial^2 L}{\partial q_1 \partial q_n} \\ \frac{\partial^2 L}{\partial q_2 \partial q_1} & \frac{\partial^2 L}{\partial q_2^2} & \cdots & \frac{\partial^2 L}{\partial q_2 \partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial q_n \partial q_1} & \frac{\partial^2 L}{\partial q_n \partial q_2} & \cdots & \frac{\partial^2 L}{\partial q_n^2} \end{pmatrix}_0 \quad (\text{A.2.39})$$

Since $(\bar{q}^0, \bar{\theta})$ corresponds to a equilibrium state

$$\left(\frac{\partial L}{\partial \bar{q}}\right)_0 = \left(\frac{\partial T}{\partial \bar{q}}\right)_0 - \left(\frac{\partial V}{\partial \bar{q}}\right)_0 = \left(\frac{\partial T_0}{\partial \bar{q}}\right)_0 - \left(\frac{\partial V}{\partial \bar{q}}\right)_0 = -\left(\frac{\partial V^*}{\partial \bar{q}}\right)_0 = 0 \quad (\text{A.3.40})$$

where we assume that (A.2.38) is satisfied. Furthermore

$$\left(\frac{\partial^2 L}{\partial \bar{q}^2}\right)_0 = \left(\frac{\partial^2 T_0}{\partial \bar{q}^2}\right)_0 - \left(\frac{\partial^2 V}{\partial \bar{q}^2}\right)_0 = -\left(\frac{\partial^2 V^*}{\partial \bar{q}^2}\right)_0, \quad \left(\frac{\partial^2 L}{\partial \bar{q} \partial \dot{\bar{q}}}\right)_0 = \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}}\right)_0 \quad \text{and} \quad \left(\frac{\partial^2 L}{\partial \dot{\bar{q}}^2}\right)_0 = \left(\frac{\partial^2 T_2}{\partial \dot{\bar{q}}^2}\right)_0 \quad (\text{A.2.41})$$

and then

$$L(t, \bar{q}, \dot{\bar{q}}) = L(t, \bar{q}^0, \bar{\theta}) + \left(\frac{\partial T_1}{\partial \dot{\bar{q}}}\right)_0 \dot{\bar{r}} - \frac{1}{2} \bar{r}^T \left(\frac{\partial^2 V^*}{\partial \bar{q}^2}\right)_0 \bar{r} + \bar{r}^T \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}}\right)_0 \dot{\bar{r}} + \frac{1}{2} \dot{\bar{r}}^T \left(\frac{\partial^2 T_2}{\partial \dot{\bar{q}}^2}\right)_0 \dot{\bar{r}} + \dots \quad (\text{A.2.42})$$

We now use coordinates $(\bar{r}, \dot{\bar{r}})$ instead of $(\bar{q}, \dot{\bar{q}})$ and calculate the partial derivatives

$$\frac{\partial L}{\partial \dot{\bar{r}}} = \left(\frac{\partial T_1}{\partial \dot{\bar{q}}}\right)_0 + \bar{r}^T \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}}\right)_0 + \dot{\bar{r}}^T \left(\frac{\partial^2 T_2}{\partial \dot{\bar{q}}^2}\right)_0 + \dots \quad (\text{A.2.43})$$

$$\frac{\partial L}{\partial \bar{r}} = -\bar{r}^T \left(\frac{\partial^2 V^*}{\partial \bar{q}^2}\right)_0 + \dot{\bar{r}}^T \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}}\right)_0^T + \dots \quad (\text{A.2.44})$$

For the internal force

$$\bar{Q}^{int} = (Q_1^{int} \quad Q_2^{int} \quad \dots \quad Q_n^{int}) \quad (\text{A.2.45})$$

we obtain

$$\bar{Q}^{int}(t, \bar{q}, \dot{\bar{q}}) = \bar{Q}^{int}(t, \bar{q}^0 + \bar{r}, \dot{\bar{r}}) = \bar{Q}^{int}(t, \bar{q}^0, \bar{0}) + \bar{r}^T \left(\frac{\partial \bar{Q}^{int}}{\partial \bar{q}} \right)_0 + \dot{\bar{r}}^T \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{\bar{q}}} \right)_0 + \dots = \dot{\bar{r}}^T \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{\bar{q}}} \right)_0 + \dots \quad (\text{A.2.46})$$

if we assume that

$$\bar{Q}^{int}(t, \bar{q}^0, \bar{0}) = \bar{0} \quad \text{and} \quad \left(\frac{\partial \bar{Q}^{int}}{\partial \bar{q}} \right)_0 = \underline{0} \quad (\text{A.2.47})$$

Note that the matrix $\left(\frac{\partial \bar{Q}^{int}}{\partial \bar{q}} \right)_0$ is, in general un-symmetric. It is defined by

$$\left(\frac{\partial \bar{Q}^{int}}{\partial \bar{q}} \right)_0 = \begin{pmatrix} \frac{\partial Q_1^{int}}{\partial q_1} & \frac{\partial Q_2^{int}}{\partial q_1} & \dots & \frac{\partial Q_n^{int}}{\partial q_1} \\ \frac{\partial Q_1^{int}}{\partial q_2} & \frac{\partial Q_2^{int}}{\partial q_2} & \dots & \frac{\partial Q_n^{int}}{\partial q_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial Q_1^{int}}{\partial q_n} & \frac{\partial Q_2^{int}}{\partial q_n} & \dots & \frac{\partial Q_n^{int}}{\partial q_n} \end{pmatrix}_0 \quad (\text{A.2.48})$$

Lagrange's equations may now be written

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}_i} \right) - \frac{\partial L}{\partial r_i} = \dot{\bar{r}}^T \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}} \right)_0 + \ddot{\bar{r}}^T \left(\frac{\partial^2 T_2}{\partial^2 \dot{\bar{q}}} \right)_0 + \\ &+ \bar{r}^T \left(\frac{\partial^2 V^*}{\partial^2 \bar{q}} \right)_0 - \dot{\bar{r}}^T \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}} \right)_0^T - \dot{\bar{r}}^T \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{\bar{q}}} \right)_0 = Q_i^{ext} + \dots, \quad i = 1, \dots, n \end{aligned} \quad (\text{A.2.49})$$

Transposing and re-arranging this equation will leave us with

$$\left(\frac{\partial^2 T_2}{\partial^2 \dot{\bar{q}}} \right)_0 \ddot{\bar{r}} + \left(\left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}} \right)_0^T - \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{\bar{q}}} \right)_0 - \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{\bar{q}}} \right)_0^T \right) \dot{\bar{r}} + \left(\frac{\partial^2 V^*}{\partial^2 \bar{q}} \right)_0 \bar{r} = Q_i^{ext} + \dots, \quad i = 1, \dots, n \quad (\text{A.2.50})$$

where we have used the symmetry

$$\left(\frac{\partial^2 T_2}{\partial^2 \dot{\bar{q}}} \right)_0^T = \left(\frac{\partial^2 T_2}{\partial^2 \dot{\bar{q}}} \right)_0 \quad (\text{A.2.51})$$

We now define the following matrices: The *mass matrix*

$$\underline{M} = \left(\frac{\partial^2 T_2}{\partial^2 \dot{q}} \right)_0, \quad \underline{M}^T = \underline{M} \quad (\text{A.2.52})$$

the '*B-matrix*'

$$\underline{B} = \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{q}} \right)_0^T - \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{q}} \right)_0 - \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} \right)_0^T \quad (\text{A.2.53})$$

an the *stiffness matrix*

$$\underline{K} = \left(\frac{\partial^2 V^*}{\partial^2 \bar{q}} \right)_0, \quad \underline{K}^T = \underline{K} \quad (\text{A.2.54})$$

If we assume that the q -coordinate system is regular then we may conclude that the mass matrix is positive definite, i.e.

$$\bar{u}^T \underline{M} \bar{u} > 0, \quad \forall \bar{u} \neq \bar{0} \in \mathbb{R}^n \quad (\text{A.2.55})$$

and then $\underline{M}$ is invertible, i.e. $\underline{M}^{-1}$ exists. Furthermore, if the modified potential has a local minimum at the equilibrium state then the stiffness matrix is *positive semi-definite*, i.e.

$$\bar{u}^T \underline{K} \bar{u} \geq 0, \quad \forall \bar{u} \neq \bar{0} \in \mathbb{R}^n \quad (\text{A.2.56})$$

The B -matrix is in general not symmetric. A decomposition of $\underline{B}$ into its symmetric and anti-symmetric parts gives

$$\underline{B} = \underline{C} + \underline{G} \quad (\text{A.2.57})$$

where

$$\underline{C} = \frac{1}{2}(\underline{B} + \underline{B}^T) = -\frac{1}{2} \left(\left(\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} \right)_0 + \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} \right)_0^T \right), \quad \underline{C}^T = \underline{C} \quad (\text{A.2.58})$$

is the so-called *damping matrix* and

$$\begin{aligned} \underline{G} &= \frac{1}{2}(\underline{B} - \underline{B}^T) = -\frac{1}{2} \left(\left(\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} \right)_0 - \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} \right)_0^T \right) - \left(\left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{q}} \right)_0 - \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{q}} \right)_0^T \right) = \\ &= -\frac{1}{2} \left(\left(\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} \right)_0 - \left(\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} \right)_0^T \right) + (\tilde{g}_{ij}), \quad \underline{G}^T = -\underline{G} \end{aligned} \quad (\text{A.2.59})$$

is the *gyroscopic matrix*. We have here used the fact that

$$\left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{q}} \right)_0 - \left(\frac{\partial^2 T_1}{\partial \bar{q} \partial \dot{q}} \right)_0^T = (\tilde{g}_{ij})^T = -(\tilde{g}_{ij}) \quad (\text{A.2.60})$$

Note that if

$$\bar{Q}^{int} = -\frac{\partial D}{\partial \dot{q}} \quad (\text{A.2.61})$$

where $D = D(t, q, \dot{q})$ is a dissipation function then

$$\frac{\partial \bar{Q}^{int}}{\partial \dot{q}} = -\frac{\partial D}{\partial \dot{q}^2} \quad (\text{A.2.62})$$

and this is a symmetric matrix and consequently

$$\underline{C} = -\left(\frac{\partial D}{\partial \dot{q}^2}\right)_0, \quad \underline{G} = (\tilde{g}_{ij}) \quad (\text{A.2.64})$$

The anti-symmetry of $\underline{G}$ has as a consequence

$$\bar{u}^T \underline{G} \bar{u} = 0, \quad \forall \bar{u} \in \mathbb{R}^n \quad (\text{A.2.65})$$

The equations of motion for *mechanical vibrations* may now be written

$$\underline{M} \ddot{\underline{r}} + \underline{B} \dot{\underline{r}} + \underline{K} \underline{r} = \bar{\underline{P}} \quad (\text{A.2.66})$$

where

$$\bar{\underline{P}} = Q_i^{extT} \quad (\text{A.2.67})$$

is the external force.

Appendix A.3 Extremes of real valued functions

Let $f = f(x, y)$ be a real-valued continuously differentiable function. The necessary condition for f to have a *stationary point* (minimum, maximum or indifferent) is

$$\frac{\partial f(x, y)}{\partial x} = 0, \quad \frac{\partial f(x, y)}{\partial y} = 0 \quad (\text{A.3.1})$$

Every solution $x = x_0, y = y_0$ to (A.2.1) is a stationary point of the function f . If

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2} > 0, \quad \frac{\partial^2 f(x_0, y_0)}{\partial y^2} > 0, \quad \frac{\partial^2 f(x_0, y_0)}{\partial x^2} \frac{\partial^2 f(x_0, y_0)}{\partial y^2} - \left(\frac{\partial^2 f(x_0, y_0)}{\partial y \partial x} \right)^2 > 0 \quad (\text{A.3.2})$$

then the stationary point is a *strict local minimum*. If

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2} < 0, \quad \frac{\partial^2 f(x_0, y_0)}{\partial y^2} < 0, \quad \frac{\partial^2 f(x_0, y_0)}{\partial x^2} \frac{\partial^2 f(x_0, y_0)}{\partial y^2} - \left(\frac{\partial^2 f(x_0, y_0)}{\partial y \partial x} \right)^2 > 0 \quad (\text{A.3.3})$$

then the stationary point is a *strict local maximum*. If

$$\frac{\partial^2 f(x_0, y_0)}{\partial x^2} \frac{\partial^2 f(x_0, y_0)}{\partial y^2} - \left(\frac{\partial^2 f(x_0, y_0)}{\partial y \partial x} \right)^2 < 0 \quad (\text{A.3.4})$$

then the stationary point is a *saddle point*.

Appendix A.4: On the definition of relative damping for complex modes

Given a mechanical system with the impedance matrix

$$\mathbf{Z}(s) = \mathbf{M}s^2 + \mathbf{C}s + \mathbf{K} \quad (\text{A.4.1})$$

Let $s = \sigma + i\omega \in \mathbb{C}$, $\sigma, \omega \in \mathbb{R}$, $\sigma < 0$, $\omega \neq 0$, be a root to the characteristic equation

$$p(s) = \det(\mathbf{Z}(s)) = 0 \quad (\text{A.4.2})$$

and let $\mathbf{z}$ be the corresponding mode shape vector, i.e. $\mathbf{Z}(s)\mathbf{z} = \mathbf{0}$, then the *relative damping* ζ^c corresponding to this mode $(s, \mathbf{z})$ is defined by

$$\zeta^c = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\sigma}\right)^2}} \quad (\text{A.4.3})$$

For a weakly damped system, which is diagonalizable using the classical normal modes, we have (see Lecture Notes on p.150)

$$\sigma = -\zeta \omega_0, \quad \omega = \omega_d = \omega_0 \sqrt{1 - \zeta^2} \quad \text{if } 0 \leq \zeta < 1 \quad (\text{A.4.4})$$

and then

$$\zeta^c = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\sigma}\right)^2}} = \frac{1}{\sqrt{1 + \left(\frac{\omega_0 \sqrt{1 - \zeta^2}}{-\zeta \omega_0}\right)^2}} = \zeta \quad (\text{A.4.5})$$

Appendix A.5: Partitioned matrices

Consider the matrix

$$\underline{A} = \begin{pmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{pmatrix} \in \mathbb{R}^{(n+m) \times (n+m)} \quad (\text{A.5.1})$$

where $\underline{A} \in \mathbb{R}^{m \times m}$, $\underline{D} \in \mathbb{R}^{n \times n}$ and $\underline{C} \in \mathbb{R}^{n \times m}$ and $\underline{B} \in \mathbb{R}^{m \times n}$. Calculate $\det \underline{A}$!

Case 1: Assume that $\det \underline{A} \neq 0$. Then

$$\det \underline{A} = \det \underline{A} \cdot \det(\underline{D} - \underline{C} \underline{A}^{-1} \underline{B}) \quad (\text{A.5.2})$$

Case 2: Assume that $\det \underline{D} \neq 0$. Then

$$\det \underline{A} = \det(\underline{A} - \underline{B} \underline{D}^{-1} \underline{C}) \cdot \det \underline{D} \quad (\text{A.5.3})$$

Note that if all matrices are square, i.e. $m = n$ then (A.5.2) and (A.5.3) may be replaced by

$$\det \underline{A} = \det(\underline{A} \underline{D} - \underline{A} \underline{C} \underline{A}^{-1} \underline{B}) \quad (\text{A.5.4})$$

and

$$\det \underline{A} = \det(\underline{A} \underline{D} - \underline{B} \underline{D}^{-1} \underline{C} \underline{D}) \quad (\text{A.5.5})$$

respectively. Now if $\underline{A}$ and $\underline{C}$ commute, i.e. $\underline{A} \underline{C} = \underline{C} \underline{A}$ then (A.5.4) may be written

$$\det \underline{A} = \det(\underline{A} \underline{D} - \underline{C} \underline{B}) \quad (\text{A.5.6})$$

and if $\underline{C}$ and $\underline{D}$ commute, i.e. $\underline{C} \underline{D} = \underline{D} \underline{C}$ then (A.5.5) may be written

$$\det \underline{A} = \det(\underline{A} \underline{D} - \underline{B} \underline{C}) \quad (\text{A.5.7})$$

Now consider the matrix

$$\begin{pmatrix} \underline{B}s + \underline{K} & \underline{M}s \\ \underline{M}s & -\underline{M} \end{pmatrix} \quad (\text{A.5.8})$$

where $\underline{M}$ is the mass matrix, $\underline{B}$ the ‘B-matrix’ and $\underline{K}$ the stiffness matrix of a mechanical system. Since $\det(-\underline{M}) \neq 0$ we obtain, by using expression (A.5.3)

$$\begin{aligned} \det \begin{pmatrix} \underline{B}s + \underline{K} & \underline{M}s \\ \underline{M}s & -\underline{M} \end{pmatrix} &= \det(\underline{B}s + \underline{K} - \underline{M}s(-\underline{M})^{-1} \underline{M}s) \cdot \det(-\underline{M}) = \\ &= \det(\underline{M}s^2 + \underline{B}s + \underline{K}) \cdot \det(-\underline{M}) = \det \underline{Z}(s) \cdot \det(-\underline{M}) = (-1)^n \det \underline{M} p(s) \end{aligned} \quad (\text{A.5.9})$$