MECHANICAL VIBRATIONS
EXERCISES

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MECHANICAL VIBRATIONS
Introductory exercises

The student may consult the basic course in mechanics for information on vibrations of one-dimensional systems, for instance the textbook Meriam & Kraige; Dynamics, Chapter 8 or Nyberg C.; Mekanik, Grundkurs, Chapter 12.

Theory

1. Show that the initial value problem for a one-dimensional system performing free vibrations

$$\begin{aligned} &m\ddot{x} + kx = 0 \\ &x(0) = x_0,\ \dot{x}(0) = v_0 \end{aligned} \tag{1}$$

where m and k are positive constants and x_0, v_0 are constants prescribing the initial conditions, has the solution

$$x(t) = x_0 \cos\omega_0 t + \frac{v_0}{\omega_0}\sin\omega_0 t \tag{2}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is called *the natural frequency* of the system.

2. Show that the initial value problem for a one-dimensional system performing forced vibrations

$$\begin{aligned} &m\ddot{x} + kx = \hat{P}\sin\omega t \\ &x(0) = x_0,\ \dot{x}(0) = v_0 \end{aligned} \tag{3}$$

where $\hat{P}$ and ω are constants characterizing the 'external driving force', has the solution

$$x(t) = x_0 \cos\omega_0 t + \frac{1}{\omega_0}(v_0 - g(\frac{\omega}{\omega_0})\frac{\hat{P}}{k}\omega)\sin\omega_0 t + g(\frac{\omega}{\omega_0})\frac{\hat{P}}{k}\sin\omega t \tag{4}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ and $g(\theta) = \frac{1}{1-\theta^2}$ is the so-called *magnification factor.*

3. Using the solution (4), in exercise 2 above, and the following data:

$$m = 1.0\,kg,\ k = 1000\,Nmm^{-1},\ x_0 = 1.0\,mm,\ v_0 = 10\,mms^{-1},$$
$$\hat{P} = 100\,N,\ \omega = 20\,rads^{-1}$$

Plot (using a math software such as Matlab or Maple) the motion $x = x(t)$

a) in the time interval $0 \le t \le 2$.
b) in the time interval $0 \le t \le 10$.

4. If viscous damping is introduced into the model then the initial value problem for the system performing forced vibrations may be written

$$\begin{aligned} &m\ddot{x} + c\dot{x} + kx = \hat{P}\sin\omega t \\ &x(0) = x_0,\ \dot{x}(0) = v_0 \end{aligned} \tag{5}$$

where c is a positive constant ('the viscous damping coefficient'). Calculate (using a math software such as Matlab or Maple) the solution to the initial value problem. Use the data in exercise 3 above along with the viscous damping coefficient $c = 3\,Nsmm^{-1}$. Plot the motion $x = x(t)$

a) in the time interval $0 \le t \le 2$.
b) in the time interval $0 \le t \le 10$.

Explain the performance of the vibration. What happens in the limit $t \to \infty$. Compare with the un-damped vibration in exercise 3!

5. The so-called *steady state solution* to (5) may be written

$$x_s(t) = \frac{\hat{P}}{k} g(\frac{\omega}{\omega_0}, \varsigma)\sin(\omega t - \delta) \tag{6}$$

where $\zeta = \dfrac{c}{2m\omega_0}$ is the so-called *relative damping*. The *magnification factor* g is given by

$$g(\theta,\varsigma) = \frac{1}{\sqrt{(1-\theta^2)^2 + 4\varsigma^2\theta^2}},\quad 0 \le \theta < \infty,\ 0 \le \varsigma < \infty \tag{7}$$

and the *phase lag angle* δ is given by

$$\delta(\theta,\varsigma) = \begin{cases} arctan(\dfrac{2\varsigma\theta}{1-\theta^2}) \text{ if } 0 \le \theta \le 1 \\ arctan(\dfrac{2\varsigma\theta}{1-\theta^2}) + \pi \text{ if } \theta > 1 \end{cases} \tag{8}$$

a) Show that the steady state solution (6) is a solution to the differential equation in (5). Does it fulfil the initial conditions?

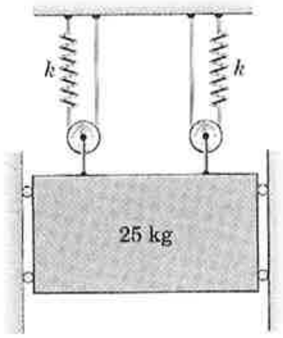
b) Plot the solution to the initial value problem (5) $x = x(t)$, as obtained in exercise 4, and the steady state solution $x_s = x_s(t)$ in the same diagram for the time interval $0 \le t \le 10$. How do the solutions differ? What happens in the long run?

c) Plot the magnification factors $g = g(\theta, 0.05)$, $0 \le \theta \le 4$ and $g = g(\theta, 0.5)$, $0 \le \theta \le 4$ in the same diagram.

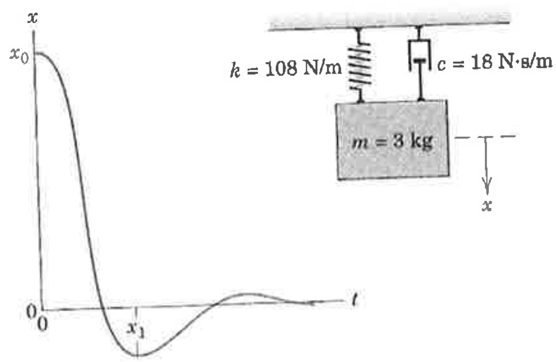
d) Plot the phase lag angles $\delta = \delta(\theta, 0.01)$, $0 \le \theta \le 4$ and $\delta = \delta(\theta, 0.1)$, $0 \le \theta \le 4$ in the same diagram.

Applications

6. Calculate the natural frequency of vertical oscillations of the 25-kg block when it is set in motion. Each spring has a stiffness of $1200\ Nm^{-1}$. Neglect the mass of the pulleys. (Meriam & Kraige, Dynamics (5th ed.), Problem 8/23).



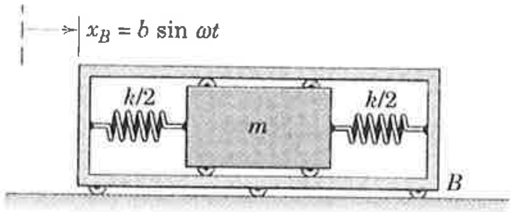
7. The system shown is released from rest from an initial position x_0. Determine the 'overshoot' displacement x_1. Assume translational motion in the x-direction. (Meriam & Kraige, Dynamics (5th ed.), Problem 8/38).



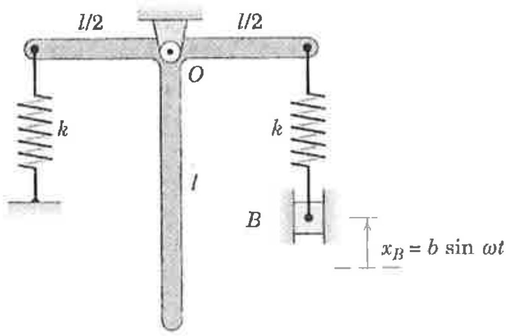
8. The motion of the outer cart B is given by

$$x_B = x_B(t) = b \sin \omega t \tag{9}$$

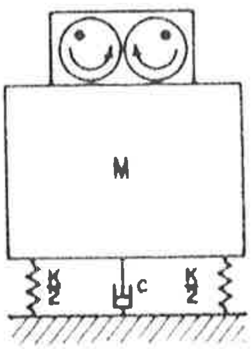
For what range of the driving frequency ω is the amplitude of the motion of the mass m relative to the cart less than $2b$. The damping of the system is assumed to be negligible. (Meriam & Kraige, Dynamics (5^{th} ed.), Problem 8/58).



9. Two identical uniform bars are welded together at right angle and are pivoted about a horizontal axis through the point shown. Determine the critical driving frequency of the block B which will result in excessively large oscillations of the assembly. The mass of the welded assembly is m. (Meriam & Kraige, Dynamics(5^{th} ed.), Problem 8/92).



10. (Rotating unbalance). A conterrotating eccentric weight exciter is used to produce forced oscillations of a spring-damper supported mass, as shown in the figure below. By varying the speed of rotation a resonant amplitude of $0.6\,cm$ was recorded. When the speed of rotation was increased considerably beyond the resonant frequency, the amplitude appears to approach a fixed value of $0.08\,cm$. Determine the *relative damping* of the system.



MECHANICAL VIBRATIONS
Exercise 1: Equations of motion

Theory

1. A mechanical system (body) $\mathcal{B}$ is subjected to the external force system $\{\mathbf{dF}^e\}$ and the internal force system $\{\mathbf{dF}^i\}$. Show that

$$\begin{cases} \mathbf{dF}^e + \mathbf{dF}^i = \mathbf{a}dm \\ \int_{\mathcal{B}} \mathbf{dF}^i = \mathbf{0} \\ \int_{\mathcal{B}} \mathbf{r}\times\mathbf{dF}^i = \mathbf{0} \end{cases} \quad \Leftrightarrow \quad \begin{cases} \mathbf{dF}^e + \mathbf{dF}^i = \mathbf{a}dm \\ \int_{\mathcal{B}} \mathbf{dF}^e = \int_{\mathcal{B}} \mathbf{a}dm \\ \int_{\mathcal{B}} \mathbf{r}\times\mathbf{dF}^e = \int_{\mathcal{B}} \mathbf{r}\times\mathbf{a}dm \end{cases} \tag{1}$$

2. Consider a mechanical system $\mathcal{B}$ whose configuration space is 1-dimensional, i.e. the position vector of a material point $P\in\mathcal{B}$ is given by $\mathbf{r}=\mathbf{r}(q;P)$ where q is *one* scalar variable, $q=q(t)$. We assume that the system has a potential energy $U=U(q)$ and that $Q^{ext}=Q^{int}=0$.

   a) Show that the kinetic energy of the system may be written

   $$T(q,\dot{q})=\frac{1}{2}a(q)\dot{q}^2 \text{ where } a(q)=\int_{\mathcal{B}}\frac{\partial\mathbf{r}}{\partial q}\cdot\frac{\partial\mathbf{r}}{\partial q}dm \tag{2}$$

   b) With the Lagrangian $L=L(q,\dot{q})=T-U$ calculate the equation of motion for the system.

   c) Show that the mechanical energy $E=E(q,\dot{q})=T+U$ is a constant (*an integral of motion*). Use this to show that a solution $q=q(t),\ t\geq 0$ to the equation of motion with the initial condition $q(0)=q_o,\ \dot{q}(0)=v_o>0$ must (at least in the first phase of the motion) satisfy

   $$t=\int_{q_o}^{q(t)}\frac{\sqrt{a(x)}}{\sqrt{2(E-U(x))}}dx \tag{3}$$

   Evaluate this relation for the case $a(q)=m,\ U(q)=\frac{k}{2}q^2$ where m and k are positive constants.

   d) Show that $q=q_o,\ \dot{q}=0$ is an equilibrium state if and only if

$$\frac{dU(q_o)}{dq} = 0 \tag{4}$$

e) Show that an equilibrium position is stable if

$$\frac{d^2U(q_o)}{dq^2} > 0 \tag{5}$$

3. A mechanical system has the Lagrangian $L = L(t, q, \dot{q})$ and the generalized forces $Q_i = Q_i(t, q, \dot{q})$. Show that

$$Q_k \equiv 0,\ \frac{\partial L}{\partial q_k} \equiv 0 \Rightarrow \frac{\partial L}{\partial \dot{q}_k} = const. \tag{6}$$

In this case we say that $\frac{\partial L}{\partial \dot{q}_k}$ is *an integral of motion.*

4. Given a mechanical system with Lagrangian function $L = L(t, q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n)$ and with the generalized mechanical energy $E = E(t, q, \dot{q}) = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$. Show that in a change of coordinates $q \to q^*$, defined by $q = q(t, q^*)$, we have the relation

$$E^* = E - \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial t} \tag{7}$$

5. A mechanical system has the Lagrangian $L = L(t, q, \dot{q})$ and the gyroscopic inertia force $K_i^g = \sum_{j=1}^{n} \tilde{g}_{ij}\dot{q}_j$. The equations of motion are linearized at an equilibrium state $q = q_o$, $\dot{q} = 0$. Show that the gyroscopic matrix $\mathbf{G} = \left[g_{ij}\right]$ may be written

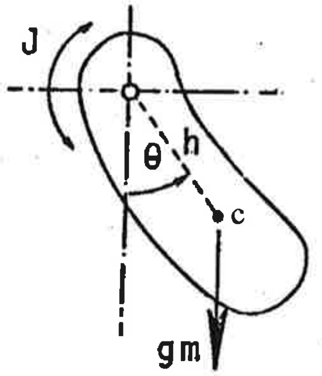
$$g_{ij} = \left(\frac{\partial^2 L}{\partial \dot{q}_i \partial q_j}\right)_o - \left(\frac{\partial^2 L}{\partial \dot{q}_j \partial q_i}\right)_o - (\tilde{g}_{ij})_o \tag{8}$$

Applications

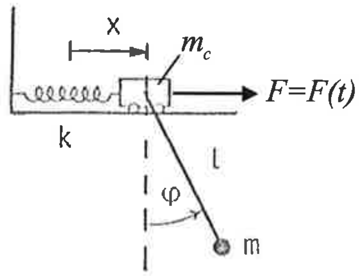
6. A physical pendulum in the form of a rigid body with mass m may rotate around a fixed horizontal axis, with respect to which its moment of inertia is J. The centre of mass c has the perpendicular distance $h > 0$ to the axis, see figure below. Use Lagrange's method to formulate the equation of motion for the system:

   a) if there is no friction in the revolute joint between body and axis.
   b) if there is a constant friction moment M_f in the revolute joint between body and axis.

   Calculate the possible equilibrium configurations of the system. Are they Liapunov stable (asymptotically stable) or unstable?

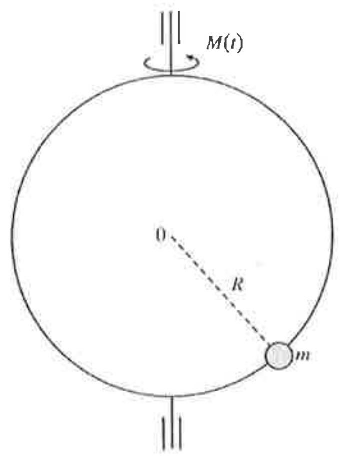


7. The pendulum system, moving in a vertical plane, consists of a small ball with mass m fixed with a string, of length l, to a carriage which may move, without friction, along a horizontal line as shown in the figure below. The carriage, with mass m_c, is connected to a fixed point with a linear elastic spring, with spring constant k, and subjected to an external force $F = F(t)$ along its line of motion. Use Lagrange's method to formulate the equations of motion for the system, ($x = 0$ corresponds to unstressed spring).



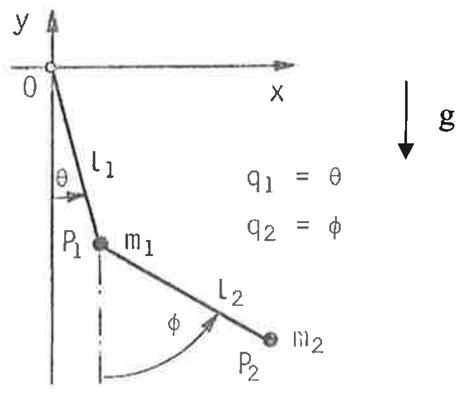
8. A mechanical system consists of a thin hoop of radius R and mass m_h that may rotate without friction around a vertical axis. A bead of mass m can slide freely (no friction) around the hoop. The hoop is acted upon by the torque $M = M(t)$ about the vertical axis.

a) Use Lagrange´s method to formulate the equations of motion for the system.
b) Determine whether there are any integrals of motion.
c) Assume that the hoop is rotating with the constant angular velocity Ω. Determine the possible equilibrium states of the system.



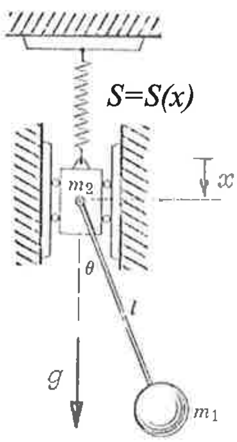
9. A double pendulum consists of two particles P_1 and P_2 with masses m_1 and m_2 respectively, connected by two strings of lengths l_1 and l_2 according to the figure below.

a) Use Lagrange´s method to formulate the equations of motion for the system.
b) Linearize the equations of motions at the equilibrium state: $\theta = \phi = 0,\ \dot{\theta} = \dot{\phi} = 0$.



10. A mechanical system consists of a particle pendulum with mass m_1 and cord-length l connected to a carriage, with mass m_2, which may move along a vertical guide, see figure below. The carriage is supported by a non-linear spring element with a retracting spring force equal to $S = S(x) = k_1 x + k_2 x^3$, $k_1, k_2 > 0$ where x is the elongation of the spring (measured from the unstressed state). The contact between the carriage and the guide walls gives rise to a constant frictional force $F_f > 0$ opposing the motion. Using the generalized coordinates x and θ

a) Formulate the equations of motion for the system.
b) Calculate the dissipation function for the system.
c) Find the equilibrium states and decide on their stability.



MECHANICAL VIBRATIONS
Exercise 2: Un-damped Vibrations

Theory

1. The free motion of an n-dimensional, linearized, mechanical system with mass-matrix $\mathbf{M}$, damping/gyroscopic-matrix $\mathbf{B}$ and stiffness-matrix $\mathbf{K}$ is given by

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{B}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{1}$$

Assume that $\mathbf{q} = \mathbf{q}(t) = \mathbf{x}e^{st}$, $s \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{R}^n$ a constant vector, is a solution to this differential equation. Show that

$$\mathbf{Z}(s)\mathbf{x} = 0 \tag{2}$$

where $\mathbf{Z}(s) = \mathbf{M}s^2 + \mathbf{B}s + \mathbf{K}$ is the so-called mechanical impedance matrix.

2. Show that if $\det\mathbf{K} = 0$ then there is a constant vector $\mathbf{u} \neq \mathbf{0}$ such that the motion ("rigid-body mode") $\mathbf{q} = \mathbf{q}(t) = \mathbf{u}\cdot(\alpha t + \beta)$, α and β constants, is a solution to

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \tag{3}$$

3. Show that the so-called Rayleigh quotient $\mathcal{R}(\mathbf{x}) = \dfrac{\mathbf{x}^T\mathbf{K}\mathbf{x}}{\mathbf{x}^T\mathbf{M}\mathbf{x}}$ for an n-dimensional mechanical system with mass-matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ satisfies

$$\omega_1^2 \leq \mathcal{R}(\mathbf{x}) \leq \omega_n^2, \quad \forall \mathbf{x} \in \mathbb{R}^n \tag{4}$$

where ω_1^2 is the smallest eigenfrequency and ω_n^2 the largest, ($\omega_1^2 \leq \omega_2^2 \leq \ldots \leq \omega_n^2$).

4. An n-dimensional linearized, undamped, non-gyroscopic mechanical system with mass-matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ has the characteristic equation

$$\det(-\omega^2\mathbf{M} + \mathbf{K}) = c_n(\omega^2)^n + c_{n-1}(\omega^2)^{n-1} + \ldots + c_0 = 0 \tag{5}$$

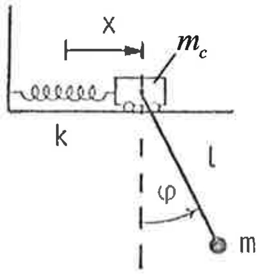
   a) Show that $c_n = (-1)^n \det\mathbf{M}$, $\quad c_0 = \det\mathbf{K}$
   b) Show that if $\det\mathbf{K} = 0$ then $\omega^2 = 0$ is a root to the characteristic equation.
   c) Show that if $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$ are the roots to the characteristic equation and $\det\mathbf{M} \neq 0$ then for the product of all roots we have

$$\omega_1^2\omega_2^2\ldots\omega_n^2 = \frac{\det\mathbf{K}}{\det\mathbf{M}} \tag{6}$$

Applications

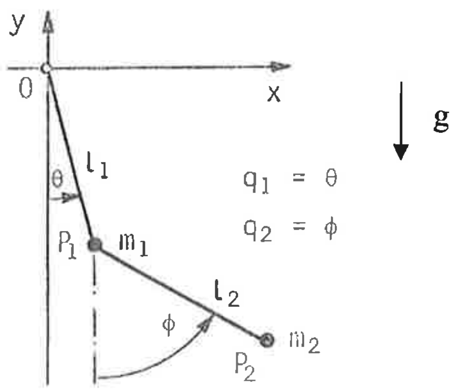
5. See Exercise 1:7. The pendulum system, moving in a vertical plane, consists of a small ball with mass m fixed with a string, of length l, to a carriage which may move, without friction, along a horizontal line as shown in the figure below. The carriage, with mass m_c, is connected to a fixed point with a linear elastic spring, with spring constant k ($x = 0$ corresponds to the unstressed spring).

    a) Formulate the linearized equations of free vibrations for the system around the equilibrium state: $\varphi = x = 0,\ \dot{\varphi} = \dot{x} = 0$.
    b) Calculate the system eigenfrequencies and the corresponding modes shapes for the case $m = m_c = 1.0\text{kg},\ k = 100\text{N/m},\ l = 1.0\text{m},\ g = 9.81\text{ms}^{-2}$.
    c) Show that the mode shapes are orthogonal with respect to the mass-matrix.
    d) Introduce the external force as in Exercise 1:7; $F = F_0 \sin(\omega t)$ and calculate a forced response (particulate solution).



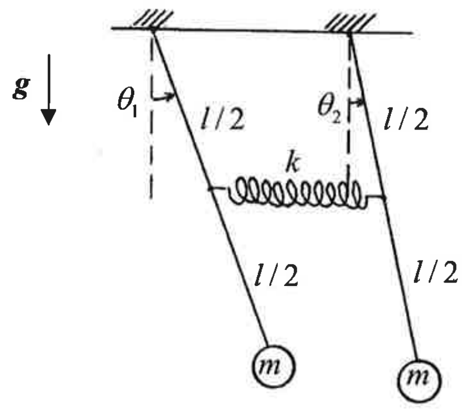
6. See Exercise 1:9. A double pendulum consists of two particles P_1 and P_2 with masses $m_1 = m_2 = m$ respectively, connected by two strings of lengths $l_1 = l_2 = l$ according to the figure below.

    a) Formulate the linearized equations of free vibrations for the system around the equilibrium state: $\theta = \phi = 0,\ \dot{\theta} = \dot{\phi} = 0$.
    b) Calculate the system eigenfrequencies and the corresponding modes shapes.



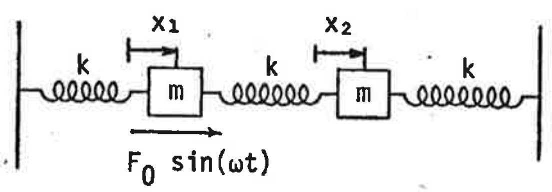
7. Consider a mechanical system consisting of two identical particle pendulums with mass $m = 0.5kg$ and cord-length $l = 2.0m$ (meter) connected with a spring whose unstressed length is equal to the distance between the points of suspension of the pendulums and with a spring-constant = k. Denote by θ_1 and θ_2 the angles of inclination of the pendulums, see figure below.

a) Formulate the linearized equations of free vibrations for the system around the equilibrium state: $\theta_1 = \theta_2 = 0,\ \dot{\theta}_1 = \dot{\theta}_2 = 0$.
b) Calculate the system eigenfrequencies and the corresponding modes shapes.
c) Suppose that the pendulums are at rest at the initial moment ($t = 0$), and one of them is given the initial angular velocity $\dot{\theta}_1(0) = \omega = 1 rad/s$. Calculate the subsequent motion in the cases $k = 10$N/m and $k = 0.1$N/m respectively. Use some math code (Matlab, Mathcad, Maple, ...) to plot the motion in a diagram.



8. Consider the mechanical system in the figure below. The masses move along a fixed horizontal line and the left mass is subjected to an external harmonic force with amplitude F_0 and angular frequency ω.

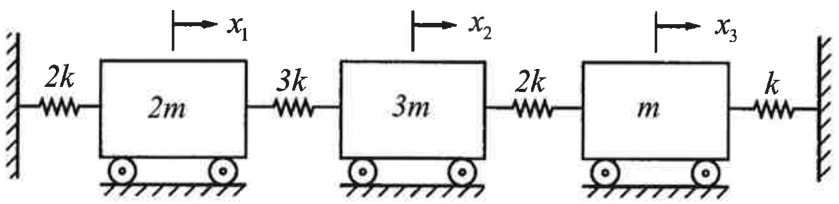
a) Use the coordinates x_1 and x_2 and formulate the equations of motion for the system.
b) Calculate the natural frequencies and mode shapes for the free vibrations ($F_0 = 0$).
c) Calculate the forced vibration ($F_0 > 0$). Is there any frequency ω for which the left mass is at rest ($x_1(t) = 0$)?



9. Consider a mechanical system consisting of three wagons with masses 2m, 3m and m respectively. The wagons are connected to each other and to the walls with springs of stiffness *2k*, *3k*, *2k* and *k* respectively, see figure below.

a) Use the coordinates x_1, x_2 and x_3 and formulate the equations of motion for the system.
b) Calculate the natural frequencies and mode shapes for the free vibrations.
c) Show that the mode shapes are orthogonal with respect to the mass- and stiffness-matrices.
d) Calculate the component $a_{11} = a_{11}(\omega)$ of the admittance matrix. Plot this function in a diagram. Find frequencies ω^a satisfying: $a_{11}(\omega^a) = 0$ (anti-resonances).
e) Introduce the constraint $x_1 \equiv 0$ (i.e. we fix the wagon on the left). Calculate the natural frequencies for this constrained system (with two degrees of freedom).

Use $\frac{k}{m} = 10^4 \text{s}^{-2}$ and $m = 1\text{kg}$ in the calculations.



MECHANICAL VIBRATIONS
Exercise 3: Damped Vibrations

Theory

1. A mechanical system is defined by its mass-matrix $\mathbf{M}$, its stiffness-matrix $\mathbf{K}$ and its damping matrix:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}, \quad \alpha, \beta \text{ (positive) real constants} \tag{1}$$

Show that

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C} \tag{2}$$

2. Let ς denote the relative modal damping and ω_0 the un-damped modal natural frequency of the system in exercise 3:1 above with $\alpha = 1.0$ and $\beta = 0.001$. Plot the function: $\varsigma = \varsigma(\omega_0), \quad 10 \le \omega_0 \le 100$.

3. The free vibrations of the system in exercise 3:1 above are given by the solution to the initial value problem:

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0} \\ &\mathbf{q}(0) = \mathbf{q}_0, \; \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \\ &\text{where } \mathbf{q}_0, \dot{\mathbf{q}}_0 \text{ are given constant vectors.} \end{aligned} \tag{3}$$

Show that the solution to this problem is given by:

$$\begin{aligned} \mathbf{q}(t) = &\left[\sum_{i=1}^{n} e^{-\varsigma_i \omega_{0i} t} \left(\frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i} \cos(\omega_{di} t) + \varsigma_i \frac{\omega_{0i}}{\omega_{di}} \frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i} \sin(\omega_{di} t) \right) \right] \mathbf{q}_0 + \\ &+ \left[\sum_{i=1}^{n} e^{-\varsigma_i \omega_{0i} t} \frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i \omega_{di}} \sin(\omega_{di} t) \right] \dot{\mathbf{q}}_0 \end{aligned} \tag{4}$$

where $\mathbf{x}_i$ denotes the classical mode shapes, $\mu_i, \omega_{0i}, \varsigma_i$ and ω_{di} are the modal mass, un-damped natural frequency, relative damping and damped natural frequency respectively, corresponding to the i^{th} mode. We assume that $0 \le \varsigma_i < 1$ (weak damping).

4. Let $p(s) = \det(\mathbf{M}s^2 + \mathbf{C}s + \mathbf{K})$ be the characteristic polynomial for a mechanical system.

   a) Show that if $p(s) = 0$ then $p(s^*) = 0$ (where s^* is the complex conjugate of s)
   b) Show that if p has a *double root* at $s = s_o$ $(p(s_o) = 0)$ then

$$p'(s_o) = 0, \; p''(s_o) \neq 0 \tag{5}$$

Applications

5. See Exercise 1:7. The pendulum system, moving in a vertical plane, consists of a small ball with mass m fixed with a string, of length l, to a carriage which may move, without friction, along a horizontal line as shown in the figure below. The carriage, with mass m_c, is connected to a fixed point with a linear elastic spring, with spring constant k ($x = 0$ corresponds to the unstressed spring). Introduce a Rayleigh damping to the system according to

$$\mathbf{C} = 0.7\mathbf{M} \tag{6}$$

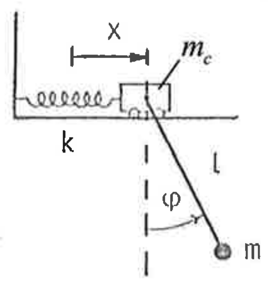
a. Formulate the equations of free vibrations for the system around the equilibrium state: $\varphi = x = 0, \; \dot{\varphi} = \dot{x} = 0$.

b. Calculate the modal relative dampings, the un-damped natural frequencies, the damped natural frequencies and the corresponding modes shapes for the case $m = m_c = 1.0\text{kg}, \; k = 100\text{N/m}, \; l = 1.0\text{m}, \; g = 9.81\text{ms}^{-2}$.

c. Show that the mode shapes are the same for the damped and the un-damped ($\mathbf{C} = \mathbf{0}$) cases.

d. Assume initial conditions $x(0) = 0, \; \varphi(0) = 0, \; \dot{x}(0) = 0, \; \dot{\varphi}(0) = 1 rad/s$. Calculate the subsequent motion in the cases $\mathbf{C} = \mathbf{0}$ and $\mathbf{C} = 0.7\mathbf{M}$ respectively. Plot the motions in a diagram.

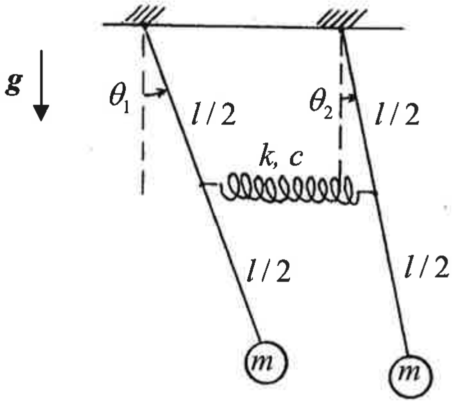
Use some math code (Matlab, Mathcad, Maple, ...) to solve the problem.



6. Consider the mechanical system in Exercise 2:7 consisting of two identical particle pendulums with mass $m = 0.5kg$ and cord-length $l = 2.0m$ (meter) connected with a spring whose unstressed length is equal to the distance between the points of suspension of the pendulums and with a spring-constant $= k = 10\text{N/m}$. Now assume that the spring has an internal viscous damping with damping constant $c = 0.5\text{Ns/m}$. Denote by θ_1 and θ_2 the angles of inclination of the pendulums, see figure below.

a. Formulate the equations of free vibrations for the system around the equilibrium state: $\theta_1 = \theta_2 = 0, \; \dot{\theta}_1 = \dot{\theta}_2 = 0$.

b. Is it possible to diagonalize this system using, for instance, the classical normal modes?



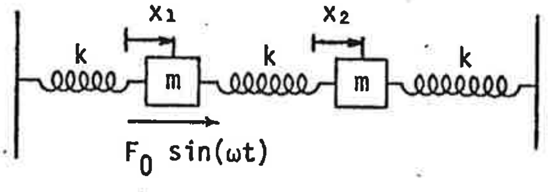
7. Consider the mechanical system in the figure below. The masses move along a fixed horizontal line and the left mass is subjected to an external harmonic force with amplitude F_0 and angular frequency ω. Introduce Rayleigh damping according to

$$\mathbf{C} = 0.006\mathbf{K} \tag{7}$$

where **C** and **K** are damping- and stiffness-matrices respectively.

a. Use the coordinates x_1 and x_2 and formulate the equations of motion for the system.
b. Calculate the modal relative dampings, the un-damped natural frequencies, the damped natural frequencies and the corresponding modes shapes for the free vibrations ($F_0 = 0$).
c. Calculate the forced vibration ($F_0 = 100\text{N}$) for the frequencies: $\omega = 30\text{rad/s}$ and $\omega = 40\text{rad/s}$ respectively. Assume that we are starting from rest, i.e $x_1(0) = x_2(0) = 0,\ \dot{x}_1(0) = \dot{x}_2(0) = 0$. Plot the position of the two masses as a function of time.
d. Calculate the forced vibration ($F_0 = 100\text{N}$) for the frequency $\omega = 30\text{rad/s}$, with the initial conditions: $x_1(0) = -0.1\text{m}, x_2(0) = 0.1\text{m},\ \dot{x}_1(0) = \dot{x}_2(0) = 0$. Plot the position of the two masses as a function of time.

Use the following data in the calculations: $m = 1\text{kg},\ k = 1000\text{N/m}$. Use some math code (Matlab, Mathcad, Maple, …) to solve the problem.



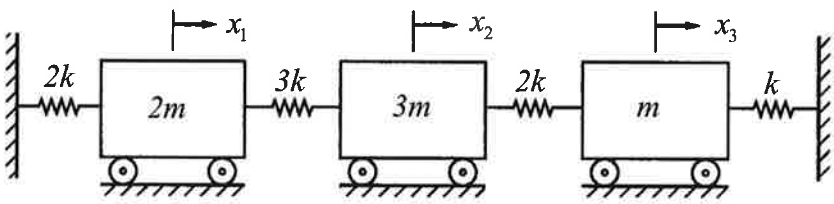
8. Consider a mechanical system consisting of three wagons with masses 2m, 3m and m respectively. The wagons are connected to each other and to the walls with springs of stiffness *2k*, *3k*, *2k* and *k* respectively, see figure below. Introduce Rayleigh damping according to

$$\mathbf{C} = 0.001\mathbf{K} \tag{8}$$

where **C** and **K** are damping- and stiffness-matrices respectively

a. Use the coordinates x_1, x_2 and x_3 and formulate the equations of motion for the system.
b. Calculate the modal relative dampings, the un-damped natural frequencies, the damped natural frequencies and the corresponding mode shapes for the free vibrations.
c. Calculate the *absolute value* of the component $a_{11} = a_{11}(i\omega)$ of the admittance matrix. Plot this function in a diagram. Are there any frequencies ω^a satisfying: $|a_{11}(\omega^a)| = 0$?

Use $\dfrac{k}{m} = 10^4\,\mathrm{s}^{-2}$ and $m = 1\mathrm{kg}$ in the calculations.



MECHANICAL VIBRATIONS
Exercise 4: Damped Vibrations

Theory

1. An n-dimensional mechanical system, with mass-matrix $\mathbf{M}$ and frequency response function $\mathbf{F} = \mathbf{F}(\omega)$ is subjected to an external force

$$\mathbf{P} = \mathbf{P}_0 \sin(\omega t) \tag{1}$$

where $\mathbf{P}_0$ is a constant amplitude vector and ω the angular velocity. The initial conditions are

$$\mathbf{q}(0) = \mathbf{q}_0,\ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \tag{2}$$

Show that if the system is "diagonizable" with natural modes:

$$\begin{matrix} \mathbf{x}_1,\dots,\mathbf{x}_n \\ s_1,\dots,s_n \end{matrix} \qquad s_i = -\zeta_i \omega_{0_i} + i\omega_{d_i},\ \ \omega_{d_i} = \omega_{0_i}\sqrt{1-\zeta_i^2},\ \ 0<\zeta_i<1 \tag{3}$$

then the motion of the system is given by:

$$\begin{aligned} \mathbf{q}(t) = &\sum_{i=1}^{n} e^{-\zeta_i \omega_{0_i} t} \frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i} (\cos(\omega_{d_i} t) + \zeta_i \frac{\omega_{0_i}}{\omega_{d_i}} \sin(\omega_{d_i} t))(\mathbf{q}_0 - \mathbf{f}(0)) + \\ &\sum_{i=1}^{n} e^{-\zeta_i \omega_{0_i} t} \frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i \omega_{d_i}} \sin(\omega_{d_i} t)(\dot{\mathbf{q}}_0 - \dot{\mathbf{f}}(0)) + \mathbf{f}(t) \end{aligned} \tag{4}$$

where

$$\mathbf{f}(t) = \mathrm{Im}(\mathbf{F}(\omega)\mathbf{P}_0 e^{i\omega t}) \tag{5}$$

Applications

2. A particle with mass m is connected to a circular ring with four linear elastic springs. The ring is rotating in a plane, around its symmetry axis perpendicular to the plane and with a constant angular velocity Ω. When the particle is positioned at the centre of the ring the springs form a perpendicular cross according to the figure below. When the particle is displaced from this position the potential energy in the spring system is given by

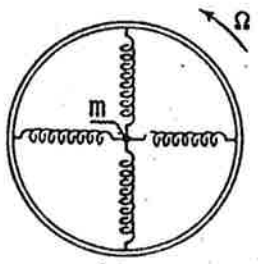
$$V = V(x,y) = V_0 + \frac{1}{2} m\omega_0^2 (x^2 + y^2) \tag{6}$$

where V_0 and ω_0 are constants and x and y denote the displacement of the particle in two perpendicular directions. It is presumed that the particle may only move in the plane of the circle.

a. Formulate the equations of motion for the system using the coordinates x and y introduced in a coordinate system rotating with the ring.
b. Show that if $\Omega^2 \neq \omega_0^2$ then $x = y = 0$ is an (relative) equilibrium state for the particle.
c. Show that the *efficient potential energy* ($V^* = V - T_0$) has a minimum at the equilibrium position if $\Omega^2 < \omega_0^2$ and a maximum if $\Omega^2 > \omega_0^2$.
d. Show that the equilibrium position is Dirichlet stable if $\Omega^2 \neq \omega_0^2$.
e. Assume that the particle is subjected to a viscous frictional force with the Rayleigh dissipation function

$$\mathcal{D}(\dot{x}, \dot{y}) = m\omega_0\zeta(\dot{x}^2 + \dot{y}^2) \tag{7}$$

where the damping constant $\zeta \ll 1$. Show that the equilibrium position is now Dirichlet stable if and only if $\Omega^2 < \omega_0^2$.



MECHANICAL VIBRATIONS
Exercise 5: Continuous systems

Theory

1. Show that the system of functions

$$\varphi_n(x)=\sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L}),\ \ 0<x<L,\ n=0,1,... \qquad (1)$$

is orthonormal, i.e.

$$\int_0^L \varphi_n(x)\varphi_m(x)dx=\begin{cases}0 & \text{if } n\neq m\\ 1 & \text{if } n=m\end{cases} \qquad (2)$$

2. The vibrational motion of *a bar in extension* is governed by the differential equation:

$$\frac{\partial}{\partial x}(EA(x)\frac{\partial u(x,t)}{\partial x})+k(x)=m(x)\frac{\partial^2 u(x,t)}{\partial t^2},\ \ 0<x<L,\ t\geq 0 \qquad (3)$$

where L is the length of the bar, E is the modulus of elasticity, $A=A(x)$ is the sectional area, $k=k(x)$ is the external force density, $m=m(x)$ the mass density, and by the boundary-initial conditions:

$$\begin{aligned}&\alpha_1 u(0,t)+\beta_1\frac{\partial u(0,t)}{\partial x}=0,\ \ \alpha_2 u(L,t)+\beta_2\frac{\partial u(L,t)}{\partial x}=0\\ &u(x,0)=u_0(x),\ \ \frac{\partial u(x,0)}{\partial t}=\dot{u}_0(x)\end{aligned} \qquad (4)$$

where $\alpha_1,\beta_1,\alpha_2,\beta_2$ are given constants ($(\alpha_1,\beta_1)\neq(0,0)$ and $(\alpha_2,\beta_2)\neq(0,0)$) and $u_0=u_0(x)$ and $\dot{u}_0=\dot{u}_0(x)$ are given functions.
Assume that $k\equiv 0$ ("free vibrations") and consider a solution on the form ("separation of variables")

$$u(x,t)=\hat{u}(x)\phi(t) \qquad (5)$$

a) show the following conditions are necessary:

$$\begin{aligned}&-\frac{1}{m(x)}\frac{d}{dx}(EA(x)\frac{d\hat{u}(x)}{dx})=\lambda\hat{u}(x)\\ &\alpha_1\hat{u}(0)+\beta_1\frac{d\hat{u}(0)}{dx}=0,\ \ \alpha_2\hat{u}(L)+\beta_2\frac{d\hat{u}(L)}{dx}=0\end{aligned} \qquad (6)$$

where λ is a constant and

$$\frac{d^2}{dt^2}\phi(t) + \lambda\phi(t) = 0 \tag{7}$$

b) Show that the differential operator ("The Sturm-Liouville operator")

$$\mathcal{L}\hat{u} = -\frac{1}{m(x)}\frac{d}{dx}(EA(x)\frac{d\hat{u}(x)}{dx}) \tag{8}$$

is symmetric, i.e. that:

$$\int_0^L (\mathcal{L}\hat{u}(x))\hat{v}(x)m(x)dx = \int_0^L (\mathcal{L}\hat{v}(x))\hat{u}(x)m(x)dx \tag{9}$$

for all $\hat{u} = \hat{u}(x)$ and $\hat{v} = \hat{v}(x)$ satisfying the boundary conditions in (6).

c) Show that if $\hat{u}_1$ and $\hat{u}_2$ are solutions to (6) then

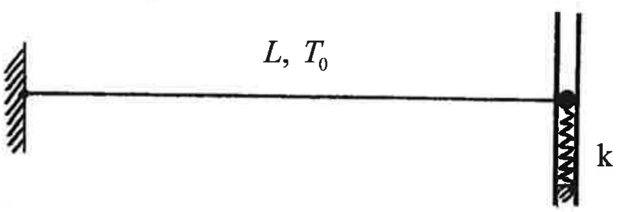
$$\lambda_1 \neq \lambda_2 \Rightarrow \int_0^L \hat{u}_1(x)\hat{u}_2(x)m(x)dx = 0 \tag{10}$$

Applications

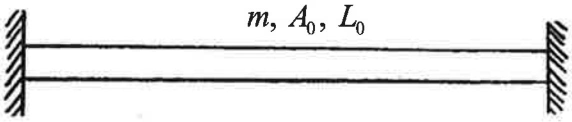
(All beams below may be modelled by the Euler-Bernoulli theory)

3. A piano string is fixed at both ends. It has a length of 1.4m and a mass of 110g. Calculate the required pre-tension T_0 so that the first natural frequency will be 424 Hz.

4. A string of steel with a circular cross section is fixed at one end and attached to a spring at the other end with a frictionless slide. See figure below! The spring constant $k = 8000\text{Nm}^{-1}$. The pre-tension of the string is $T_0 = 10.0 \cdot 10^4\,\text{N}$. Calculate the mode shapes and the natural frequencies for the string-spring system. Use the following data: string density $= \rho = 8000\text{kgm}^{-3}$, radius $r = 1.0\text{mm}$ and length $L = 1.0\text{m}$.



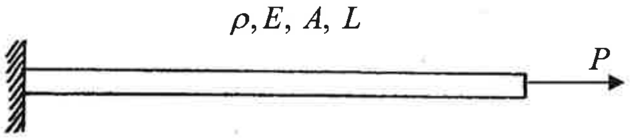
5. A homogeneous bar with total mass m, sectional area A_0 and length L_0 is clamped at both its ends. Calculate the mode shapes and natural frequencies for the system in longitudinal vibrations.



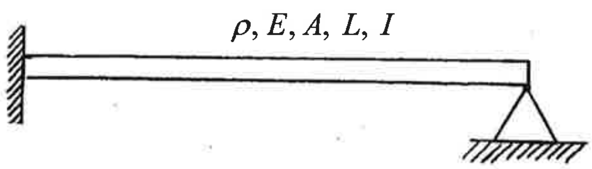
6. (Revised) A homogeneous cantilevered ("clamped-free") bar of length $L = 5.0\text{m}$ and sectional area $A = 4 \cdot 10^{-4}\text{m}^2$ is made of a material with density $\rho = 8000\text{kgm}^{-3}$ and modulus of elasticity $E = 200\text{GPa}$. The bar is initially (at time $t = 0$) at rest and is then subjected to a step loading at the "free end"

$$P(t) = \begin{cases} 0 & t < 0 \\ P_0 & t \geq 0 \end{cases}$$

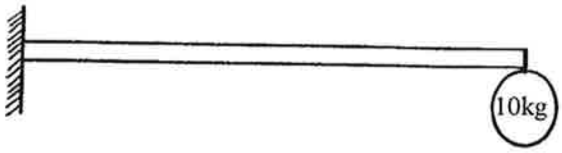
where $P_0 = 10000N$. Calculate the longitudinal response of the bar. Plot the response at the mid-point of the bar (using some math code).



7. Calculate the natural frequencies and mode shapes for the first three modes of the transverse (bending) vibrations of a beam that is clamped (fixed) at one end and simply supported at the other. The beam length $L = 1.0\text{m}$, sectional area $A = 2.6 \cdot 10^{-3}\text{m}^2$, sectional area moment-of-inertia $I = 4.7 \cdot 10^{-6}\text{m}^4$. The beam is made of a material with density $\rho = 8000\text{kgm}^{-3}$ and modulus of elasticity $E = 200\text{GPa}$.



8. Calculate the three lowest natural frequencies for the system in the figure below. The mass of the weight fixed to the end of the beam is equal to 10 kg. The beam is the same as the one specified in the previous problem 5:7. Assume that there is no influence from gravitational forces.



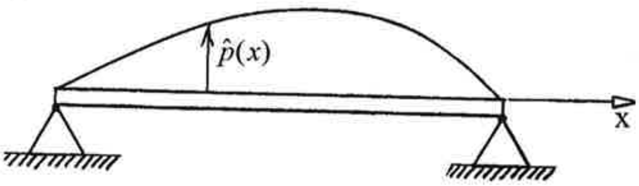
9. The simply supported beam, with density = ρ, modulus of elasticity = E, length = L, sectional area = A and sectional area moment-of-inertia = I (assumed to be constant), is subjected to an external transverse force

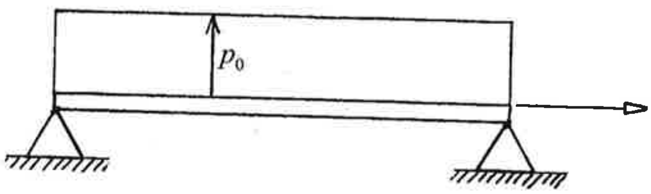
$$p = p(x,t) = \hat{p}(x)\sin \omega t \tag{11}$$

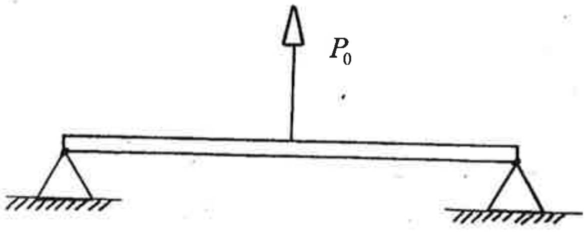
Calculate the forced "steady-state" (only consider the particulate solution, assume that the homogeneous solution is damped out in the long run) response of the midpoint $x = \frac{L}{2}$ of the beam.

a) $\hat{p}(x) = p_0 = const.$

b) $\hat{p}(x) = P_0\delta(x - \frac{L}{2})$. i.e. a point force P_0 applied at the midpoint of the beam.







MECHANICAL VIBRATIONS
Exercise 6: Continuous systems and Approximate methods

Theory

1. Let S be a surface in $\mathbb{R}^3$ and let $f = f(\mathbf{r})$ and $\mathbf{A} = \mathbf{A}(\mathbf{r})$ be a scalar- and tensor-field (2nd order) respectively defined on S. Show the calculation rule:

$$div_s(f\mathbf{A}) = f\,div_s\mathbf{A} + \mathbf{A}\nabla_s f \tag{1}$$

and use this to show that for a membrane

$$div_s(T_0\mathbf{1}_s) = T_0 2\kappa_m \mathbf{n} + \nabla_s T_0 \tag{2}$$

where T_0 is the pre-tension of the membrane, $\mathbf{1}_s$ the unit tensor on S, κ_m is the mean-curvature and $\mathbf{n} = \mathbf{n}(\mathbf{r})$ is the unit normal vector field on S.

2. Let

$$\mathbf{T}_0 = \mathbf{e}_z \otimes \mathbf{e}_x S_x + \mathbf{e}_z \otimes \mathbf{e}_y S_y \tag{3}$$

denote the *traction tensor* for a thin plate. Show that

$$div_{s_0}\mathbf{T}_0 = \mathbf{e}_z\left(\frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y}\right) \tag{4}$$

3. Let

$$\mathbf{M}_0 = -M_{xy}\mathbf{e}_x \otimes \mathbf{e}_x - M_y\mathbf{e}_x \otimes \mathbf{e}_y + M_x\mathbf{e}_y \otimes \mathbf{e}_x + M_{xy}\mathbf{e}_y \otimes \mathbf{e}_y \tag{5}$$

denote the *moment tensor* for a thin plate. Show that

$$div_{s_0}\mathbf{M}_0 = \mathbf{e}_x\left(-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y}\right) + \mathbf{e}_y\left(\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x}\right) \tag{6}$$

Applications

4. We consider a uniform rectangular membrane with mass per unit area m extending, in its un-deformed configuration, over a domain S_0 in the x-y-plane defined by

$$S_0 = \{(x,y) \in \mathbb{R}^2 : 0 < x < a,\ 0 < y < b\} \tag{7}$$

The boundary of S_0 is the "curve" Γ_0 consisting of the straight lines $x = 0,\ x = a$ and $\mathrm{y} = 0,\ \mathrm{y} = b$. The membrane is supported along Γ_0 and the pre-tension of the membrane is a constant equal to T_0

a) Using membrane theory, formulate the equations of motion for the free transversal vibrations of the membrane.
b) Formulate the boundary conditions corresponding to the membrane fixed at the boundary Γ_0.
c) Look for standing wave solutions and derive the necessary eigenvalue problems.
d) Calculate the eigenmodes and the corresponding eigenfrequencies.

5. We consider a uniform rectangular plate with mass per unit area m and thickness h extending, in its un-deformed configuration, over a domain S_0 in the x-y-plane defined by

$$S_0 = \{(x,y) \in \mathbb{R}^2 : 0 \le x \le a,\ 0 \le y \le b\} \tag{8}$$

The boundary of S_0 is the "curve" Γ_0 consisting of the straight lines $x = 0,\ x = a$ and $\mathrm{y} = 0,\ \mathrm{y} = b$. The plate is simply supported along Γ_0. The modulus of elasticity and the Poisson's ratio of the plate material are denoted E and ν respectively.

a) Using thin plate theory, formulate the equations of motion for the free transversal vibrations of the plate.
b) Formulate the boundary conditions corresponding to the simply supported plate.
c) Look for standing wave solutions and derive the necessary eigenvalue problems.
d) Calculate the eigenmodes (mode shapes) and the corresponding eigenfrequencies.

6. We consider a uniform circular plate with mass per unit area m and thickness h extending, in its un-deformed configuration, over a domain S_0 in the x-y-plane defined by

$$S_0 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le a^2\} \tag{9}$$

The boundary of S_0 is the circle

$$\Gamma_0 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = a^2\} \tag{10}$$

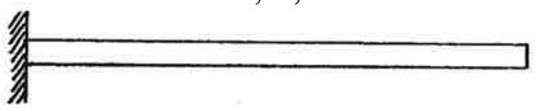
The plate is clamped along Γ_0. The modulus of elasticity and the Poisson's ratio of the plate material are denoted E and ν respectively.

a) Using thin plate theory, formulate the equations of motion for the free transversal vibrations of the plate.
b) Formulate the boundary conditions corresponding to the simply supported plate.
c) Look for standing wave solutions and derive the necessary eigenvalue problems.
d) Calculate the eigenmodes and the corresponding eigenfrequencies.

7. A homogeneous clamped-free bar of length $L = 5.0\text{m}$ is made of a material with density $\rho = 8000\text{kgm}^{-3}$ and modulus of elasticity $E = 200\text{GPa}$. Calculate the first two eigenmodes in *longitudinal vibrations* using the Rayleigh-Ritz method and the two-dimensional approximation of the displacement field:

$$\hat{u}(x) = \frac{x}{L} q_1 + \left(\frac{x}{L}\right)^2 q_2, \; 0 < x < L \tag{11}$$

where q_1 and q_2 are generalized coordinates. Compare these with the exact modes!



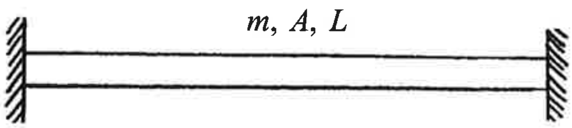
8. The homogeneous cantilevered beam in the previous exercise has a bending stiffness EI. Calculate the first two eigenfrequencies in *transversal vibrations* using the Rayleigh-Ritz method and the two-dimensional approximate displacement field

$$\hat{w}(x) = \left(\frac{x}{L}\right)^2 q_1 + \left(\frac{x}{L}\right)^3 q_2, \; 0 < x < L \tag{12}$$

where q_1 and q_2 are generalized coordinates. Compare these with the exact frequencies!

9. A homogeneous bar with *total mass* m, sectional area A and length L is clamped at both its ends. The modulus of elasticity of the bar material is denoted E. Calculate the natural frequencies for the system in *longitudinal vibrations* using the FEA with a two-element approximation and both a consistent-mass and a lumped-mass matrix.

Compare the natural frequencies of these two approximations and the exact frequencies obtained in Exercise 4:5.



10. The homogeneous beam in the previous exercise has the bending stiffness EI. Calculate the natural frequencies for the system in *transversal vibrations* using the FEA with first a two-element approximation and then a three-element approximation. Compare the natural frequencies of the two approximations with the exact frequencies.

Use the data: $EI = 9.4 \cdot 10^5\,\mathrm{Nm}^2$, $m = 10\mathrm{kg}$, $L = 2.0\mathrm{m}$

MECHANICAL VIBRATIONS
SOLUTIONS
TO EXERCISES

Autumn 2014

Division of Mechanics
Lund University
Faculty of Engineering, LTH

MECHANICAL VIBRATIONS

Introductory exercises

Problem solutions

1. The proposed solution:

$$x(t) = X_0 \cos\omega_0 t + \frac{V_0}{\omega_0}\sin\omega_0 t \qquad (1)$$

Taking time derivatives we get:

$$\dot{x}(t) = -X_0\omega_0 \sin\omega_0 t + \frac{V_0}{\omega_0}\omega_0 \cos\omega_0 t$$

$$\ddot{x}(t) = -X_0\omega_0^2 \cos\omega_0 t - \frac{V}{\omega_0}\omega_0^2 \sin\omega_0 t =$$

$$-\omega_0^2 X(t) \qquad (2)$$

$$x(0) = X_0 \cos\omega_0\cdot 0 + \frac{V_0}{\omega_0}\sin\omega_0\cdot 0 = X_0$$

$$\dot{x}(0) = -X_0\omega_0 \sin\omega_0\cdot 0 + \frac{V_0}{\omega_0}\omega_0 \cos\omega_0\cdot 0 = V_0$$

and the initial conditions år obviously fulfilled. What about the differential equation? From (2) we obtain

$$m\ddot{x} + kx = m(-\omega_0^2 x) + kx = -kx + kx =$$
$$0$$

since $m\omega_0^2 = k$. The proposed solution thus satisfies the differential equation.

#

2. The proposed solution:

$$x(t) = X_0 \cos\omega_0 t + \frac{1}{\omega_0}\left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\sin\omega_0 t +$$
$$g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\sin\omega t \qquad (1)$$

Taking the time derivatives we get:

$$\dot{x}(t) = -X_0\omega_0 \sin\omega_0 t + \frac{1}{\omega_0}\left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\omega_0 \cdot$$
$$\cos\omega_0 t + g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\cos\omega t = -X_0\omega_0\sin\omega_0 t +$$
$$\left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\cos\omega_0 t + g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\cos\omega t \qquad (2)$$

$$\ddot{x} = -X_0\omega_0^2 \cos\omega_0 t - \left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\omega_0 \sin\omega_0 t$$
$$- g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega^2\sin\omega t \qquad (3)$$

$$x(0) = x_0$$

$$\dot{x}(0) = \left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right) + g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega = V_0$$

and the initial conditions are satisfied.

From (1) and (3) we obtain:

$$m\ddot{x} + kx = m\left(-x_0\omega_0^2\cos\omega_0 t -\right.$$

$$\left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\omega_0\sin\omega_0 t - \left.g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega^2\sin\omega t\right) +$$

$$k\left(x_0\cos\omega_0 t + \frac{1}{\omega_0}\left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\sin\omega_0 t +\right.$$

$$\left.g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\sin\omega t\right) = \left(-x_0 m\omega_0^2 + kx_0\right)\cos\omega_0 t +$$

$$\left\{\left(-m\left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\omega_0 + \frac{k}{\omega_0}\left(V_0 - g\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega\right)\right)\right\}\sin\omega_0 t$$

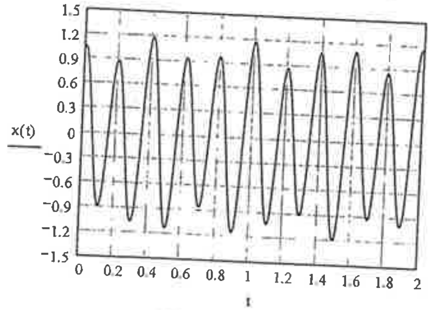
$$+\left(-mg\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\omega^2 + kg\left(\frac{\omega}{\omega_0}\right)\frac{\hat{P}}{k}\right)\sin\omega t =$$

$$\hat{P}\sin\omega t$$

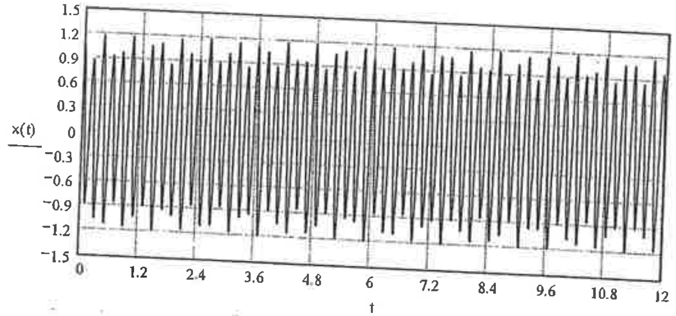
and the differential equations is satisfied.

---#

3. a)



b)



Why do we have the local irregularities in these curve?

---#

4. The second order ordinary differential equation:

$$m\ddot{x} + c\dot{x} + kx = \hat{P}\sin\omega t \qquad (1)$$

may be transformed to a system of first order differential by introducing the two unknowns:

$$x_1 = x\,, \quad x_2 = \dot{x} \qquad (2)$$

Then (1) is equivalent to

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}\left(\hat{P}\sin\omega t - cx_2 - kx_1\right) \end{aligned} \qquad (3)$$

with initial conditions:

$$\begin{aligned} x_1(0) &= x(0) = x_0 = 1.0 \\ x_2(0) &= \dot{x}(0) = v_0 = 10 \end{aligned} \qquad (4)$$

Equation (3) may be solved by using som math software such as Matlab, Maple, Mathcad etc. In Matlab there

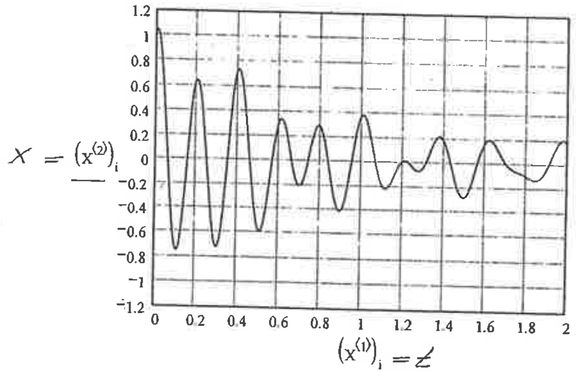
are several routines for solving systems of ordinary differential equations, for instance Ode45. Here we will use the Mathcad software.

Matcad problem definition:

$$D(t,x) := \begin{bmatrix} x_2 \\ \frac{1}{m}\cdot\left(\hat{P}\cdot\sin(\omega\cdot t) - c\cdot x_2 - k\cdot x_1\right) \end{bmatrix} \qquad x0 := \begin{pmatrix} 1 \\ 10 \end{pmatrix}$$

$X := \mathrm{rkfixed}(x0, 0, 2, 1000, D)$

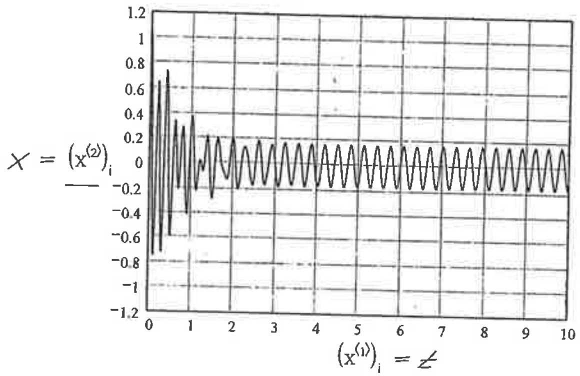
$i := 0..1000$



Why does the motion $x = x(t)$ look like this?

X := rkfixed(x0, 0, 10, 1000, D)

i := 0.. 1000



In the limit $t \to \infty$ the motion approaches a harmonic motion with constant amplitude. What is the frequency of this "steady state" motion?

——— #

5. The steady state solution:

$$x_s(t) = \frac{\hat{p}}{k} g\left(\frac{\omega}{\omega_0}, \zeta\right) \sin(\omega t - \delta)$$

a) We differentiate:

$$\dot{x}_s = \frac{\hat{p}}{k} g\left(\frac{\omega}{\omega_0}, \zeta\right) \omega \cos(\omega t - \delta)$$

$$\ddot{x}_s = -\frac{\hat{p}}{k} g\left(\frac{\omega}{\omega_0}, \zeta\right) \omega^2 \sin(\omega t - \delta)$$

Then

$$m\ddot{x}_s + c\dot{x} + kx = -m\frac{\hat{p}}{k} g\left(\frac{\omega}{\omega_0}, \zeta\right) \omega^2 \sin(\omega t - \delta)$$

$$+ c\frac{\hat{p}}{k} g\left(\frac{\omega}{\omega_0}, \zeta\right) \omega \cos(\omega t - \delta) + k\frac{\hat{p}}{k} g\left(\frac{\omega}{\omega_0}, \zeta\right).$$

$$\sin(\omega t - \delta) = \hat{p}\, g\left(\frac{\omega}{\omega_0}; \zeta\right)\left(\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) \sin(\omega t - \delta)\right.$$

$$\left. 2\zeta \frac{\omega}{\omega_0} \cos(\omega t - \delta)\right) =$$

$$\hat{p}\, g\left(\frac{\omega}{\omega_0}, \zeta\right) \sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + 4\zeta^2 \left(\frac{\omega}{\omega_0}\right)^2} \sin(\omega t - \delta$$

$+ \varphi)$, where φ is given by

$$\tan\varphi = \frac{2\zeta \frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \Rightarrow \varphi = \delta$$

Furthermore, by invoking

$$g\left(\frac{\omega}{\omega_0}, \zeta\right) = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + 4\zeta^2 \left(\frac{\omega}{\omega_0}\right)^2}}$$

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We obtain

$$m\ddot{x}_s + c\dot{x}_s + kx_s = \hat{p}\sin\omega t$$

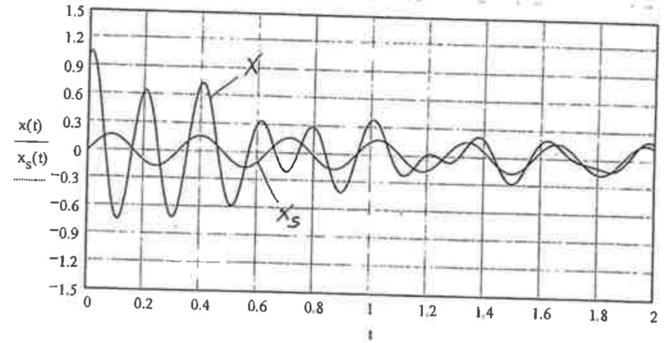
which means that the steady state solution is a solution to the differential equation. Note that

$$x_s(0) = -\frac{\hat{p}}{k}\, g\left(\frac{\omega}{\omega_0}, \zeta\right)\sin\delta$$

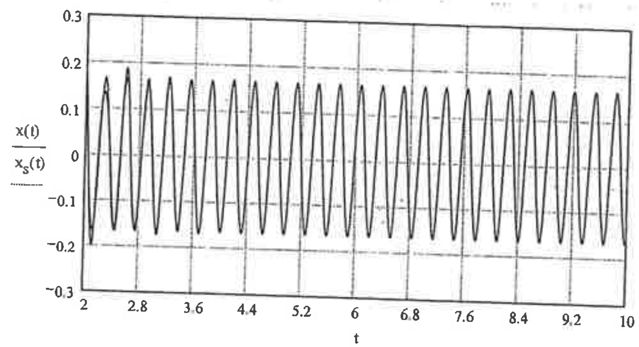
$$\dot{x}_s(0) = \frac{\hat{p}}{k}\, g\left(\frac{\omega}{\omega_0}, \zeta\right)\omega\cos\delta$$

and this is, in general, not equal to x_0 and V_0 respectively. The steady state solution <u>does not</u> fulfil the initial conditions.

b) In the timeinterval $0 \leq t \leq 2$ we have the solutions:



In the timeinterval $2 \leq t \leq 10$ we get



and the solutions become identical in the long run. How come?

c) The <u>magnification factor</u> and the <u>phase lag</u> are given in the following diagrams:

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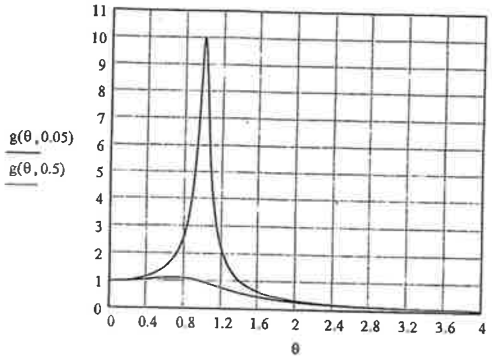
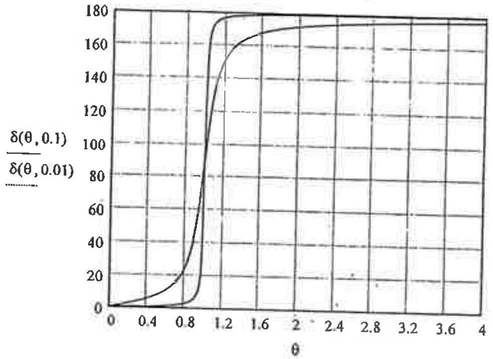


Magnification factor: $g = g(\theta, \xi)$



Phase lag: $\delta = \delta(\theta, \xi)$

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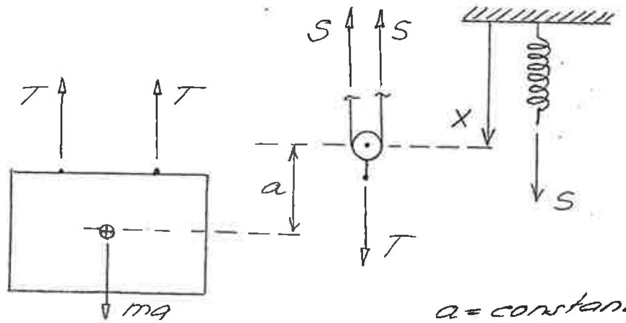
6. Draw a free-body diagram for the block.



$a =$ constant

The equation of motion for the block:

$$(\downarrow): \quad m\ddot{x} = mg - 2T \qquad (1)$$

$$T = 2S$$

Let l denote the spring length and L denote the length of the (unextendable) string. Then

$$2x + \pi r = l + L \qquad (2)$$

where r is the radius of the pulley. If l_0 is the length of the unstressed spring, (its natural length), then

$$S = k(l - l_0) = k(2x + \pi r - L - l_0) \qquad (3)$$

If (3) is inserted into (1) we get

$$m\ddot{x} + 8kx = mg - 4k(\pi r - L - l_0) \qquad (4)$$

The equilibrium position x_0 is given by:

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$$x_0 = \frac{L + l_0 - \pi r}{2} + \frac{mg}{8k}$$

Put $y = x - x_0$; the displacement from the equilibrium position, then

$$m\ddot{y} + 8ky = 0$$

$$\ddot{y} + \omega_0^2 y = 0$$

where the natural frequency:

$$\omega_0 = \sqrt{\frac{8k}{m}}$$

$k = 1200\ Nm^{-1}$, $m = 25\ kg$

$$\omega_0 = \sqrt{\frac{8 \cdot 1200}{25}} = 19.6\ s^{-1} \quad (= 3.12 Hz)$$

#

7. The natural frequency:

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{108}{3}} \cong 6\ s^{-1}$$

The relative damping:

$$\zeta = \frac{c}{2m\omega_0} = \frac{18}{2 \cdot 3 \cdot 6} = 0.5$$

The damped natural frequency:

$$\omega_d = \omega_0\sqrt{1 - \zeta^2} = 6\sqrt{1 - 0.5^2} = 5.196\ s^{-1}$$

The motion is given by

$$x(t) = e^{-\zeta\omega_0 t}(A\cos\omega_d t + B\sin\omega_d t)$$

Initial data: $x(0) = x_0$, $\dot{x}(0) = 0$

This implies the following conditions

$$A = x_0$$

$$-\zeta\omega_0 A + \omega_d B = 0 \implies B = \frac{\zeta\omega_0 x_0}{\omega_d} =$$

$0.577 x_0$

So, the motion is given by:

$$x(t) = x_0 e^{-\zeta\omega_0 t}(\cos\omega_d t + 0.577\sin\omega_d t)$$

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The period time of the motion:

$$T_d = \frac{2\pi}{\omega_d} = 1.21\,s$$

Then

$$X_1 = x\left(\frac{T_d}{2}\right) = x(0.605) = -0.1630\,x_0$$

Show that the amplitude x_1 will be obtained at $t = \frac{T}{2}$!

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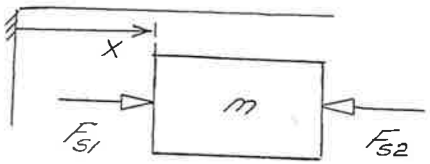
8. Draw a free-body diagram of the mass. Let x denote its displacement



relative to the cart. The equation of motion for the mass:

$$(\rightarrow):\quad m\frac{d^2}{dt^2}(x_B + x) = F_{S1} - F_{S2} \qquad (1)$$

The spring forces:

$$F_{S1} = F_0 - \frac{k}{2}x$$

$$F_{S2} = F_0 + \frac{k}{2}x$$

This gives $(x_B = b\sin\omega t)$

$$m(\ddot{x}_B + \ddot{x}) = F_0 - \frac{k}{2}x - F_0 - \frac{k}{2}x \Rightarrow$$

$$m\ddot{x} + kx = mb\omega^2\sin\omega t$$

Assume that $x(t) = X\sin\omega t$ $(X = \text{const.})$

Then

$$(-m\omega^2 + k)X\sin\omega t = mb\omega^2\sin\omega t \Rightarrow$$

$$X = \frac{\left(\frac{\omega}{\omega_0}\right)^2}{1 - \left(\frac{\omega}{\omega_0}\right)^2}\,b$$

where $\omega_0 = \sqrt{\frac{k}{m}}$

$$|X| < 2b \Leftrightarrow \quad \frac{\omega}{\omega_0} < \sqrt{\frac{2}{3}}\,,\quad \frac{\omega}{\omega_0} > \sqrt{2}$$

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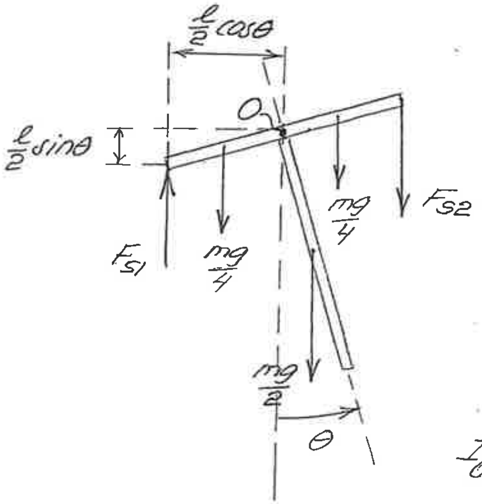
9. Draw a free-body diagram of the assembley



Spring forces:

$$F_{S1} = k\frac{\ell}{2}\sin\theta$$

$$F_{S2} = k\left(\frac{\ell}{2}\sin\theta - x_B\right)$$

$$x_B = b\sin\omega t$$

Moment of inertia:

$$I_O = \frac{1}{3}\frac{m}{2}\ell^2 + 2\left(\frac{1}{3}\frac{m}{4}\left(\frac{\ell}{2}\right)^2\right) = \frac{5}{24}m\ell^2$$

Equation of motion:

$$O:\quad -F_{S1}\frac{\ell}{2}\cos\theta + \frac{mg}{4}\frac{\ell}{4}\cos\theta - \frac{mg}{4}\frac{\ell}{4}\cos\theta - F_{S2}\frac{\ell}{2}\cos\theta - \frac{mg}{2}\frac{\ell}{2}\sin\theta = I_O\ddot{\theta}$$

Small θ: $\sin\theta \cong \theta$, $\cos\theta \cong 1$ $\Rightarrow$

$$\ddot{\theta} + \underbrace{\left(\frac{12}{5}\frac{k}{m} + \frac{6}{5}\frac{g}{\ell}\right)}_{\omega_0^2}\theta = \frac{12}{5}\frac{kb}{m\ell}\sin\omega t$$

The critical driving frequency is obtained when:

$$\omega = \omega_0 = \sqrt{\frac{6}{5}\left(\frac{2k}{m} + \frac{g}{\ell}\right)}$$

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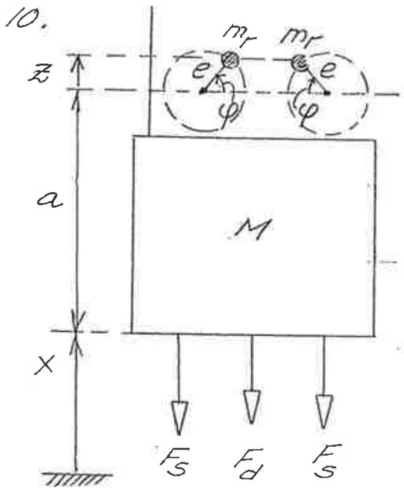
10.



Mass of eccentric weight: m_r

Spring force:

$$F_s = \frac{k}{2}x \qquad (1)$$

Damping force:

$$F_d = c\dot{x} \qquad (2)$$

Equation of motion:

$$(\uparrow):\quad M\ddot{x} + 2m_r(\ddot{x} + \ddot{z}) = -2F_s - F_d \qquad (3)$$

We assume that:

$$z = e\cos\varphi = e\sin\omega t \qquad (4)$$

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where e is the eccentricity and ω is the speed of rotation. From (1) - (4) we obtain:

$$m\ddot{x} + c\dot{x} + kx = 2m_r e\omega^2 \sin\omega t \qquad (5)$$

where

$$m = M + 2m_r \qquad (6)$$

Assume that $x(t) = X \sin(\omega t - \delta)$ solution to (5). Then (cf. Meriam & Kraige or Nyberg!) the amplitude X is given by

$$X = \frac{\frac{2m_r}{m} \cdot \left(\frac{\omega}{\omega_0}\right)^2}{\sqrt{\left(1-\left(\frac{\omega}{\omega_0}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_0}\right)^2}}\, e$$

where

$$\zeta = \frac{c}{2m\omega_0}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

$X = X(\omega)$ obtains its maximum for

$$\omega = \omega_r = \omega_0\sqrt{1-2\zeta^2} \qquad (5)$$

This is called the resonance frequency. The amplitude at resonance:

$$X_r = \frac{\frac{2m_r}{m}(1-2\zeta^2)}{2\zeta\sqrt{1-\zeta^2}}\, e = 0.6 \text{ cm}$$

$$X_\infty = \lim_{\frac{\omega}{\omega_0}\to\infty} X(\omega) = \frac{2m_r}{m} e = 0.08 \text{cm}$$

By combining these two equations we obtain;

$$\frac{0.08\,(1-2\zeta^2)}{2\zeta\sqrt{1-\zeta^2}} = 0.6 \qquad (6)$$

Assume that $\zeta \ll 1$. Then (6)

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may be written

$$\frac{0.08}{2\zeta} \cong 0.6 \Rightarrow \underline{\zeta \cong 0.0667}$$

#

MECHANICAL VIBRATIONS

Exercise 1: Equations of motion.

Problem solutions

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1. $(\Rightarrow)$: $d\bar{F}^e + d\bar{F}^i = \bar{a}\,dm \Rightarrow \int_{\mathcal{B}} d\bar{F}^e + \int_{\mathcal{B}} d\bar{F}^i = \int_{\mathcal{B}} \bar{a}\,dm$. But $\int_{\mathcal{B}} d\bar{F}^i = \bar{0}$, and then $\int_{\mathcal{B}} d\bar{F}^e = \int_{\mathcal{B}} \bar{a}\,dm$. Moreover $d\bar{F}^e + d\bar{F}^i = \bar{a}\,dm$ $\Rightarrow \int_{\mathcal{B}} \bar{r}\times d\bar{F}^e + \int_{\mathcal{B}} \bar{r}\times d\bar{F}^i = \int_{\mathcal{B}} \bar{r}\times\bar{a}\,dm$ and using $\int_{\mathcal{B}} \bar{r}\times d\bar{F}^i = \bar{0}$ we get $\int_{\mathcal{B}} \bar{r}\times d\bar{F}^e = \int_{\mathcal{B}} \bar{r}\times\bar{a}\,dm$.

$(\Leftarrow)$: $d\bar{F}^e + d\bar{F}^i = \bar{a}\,dm \Rightarrow \int_{\mathcal{B}} d\bar{F}^e + \int_{\mathcal{B}} d\bar{F}^i = \int_{\mathcal{B}} \bar{a}\,dm$. But $\int_{\mathcal{B}} d\bar{F}^e = \int_{\mathcal{B}} \bar{a}\,dm$, and then $\int_{\mathcal{B}} d\bar{F}^i = \bar{0}$. Moreover $d\bar{F}^e + d\bar{F}^i = \bar{a}\,dm \Rightarrow$ $\int_{\mathcal{B}} \bar{r}\times d\bar{F}^e + \int_{\mathcal{B}} \bar{r}\times d\bar{F}^i = \int_{\mathcal{B}} \bar{r}\times\bar{a}\,dm$. But $\int_{\mathcal{B}} \bar{r}\times d\bar{F}^e = \int_{\mathcal{B}} \bar{r}\times\bar{a}\,dm$, and then $\int_{\mathcal{B}} \bar{r}\times d\bar{F}^i = \bar{0}$.

and the equivalence is proven.

\#

2. a)

$$T = \frac{1}{2}\int_{\mathcal{B}} \dot{\bar{r}}\cdot\dot{\bar{r}}\,dm \quad , \quad \bar{r} = \bar{r}(q;P) \ , \ q = q(t)$$

$$\dot{\bar{r}} = \frac{\partial \bar{r}}{\partial q}\dot{q} \Rightarrow T = \frac{1}{2}\int_{\mathcal{B}} \frac{\partial \bar{r}}{\partial q}\cdot\frac{\partial \bar{r}}{\partial q}\dot{q}^2\,dm =$$

$$\frac{1}{2}\left(\int_{\mathcal{B}} \frac{\partial \bar{r}}{\partial q}\cdot\frac{\partial \bar{r}}{\partial q}\,dm\right)\dot{q}^2 = \frac{1}{2}a(q)\dot{q}^2$$

b) $L = L(q,\dot{q}) = \frac{1}{2}a(q)\dot{q}^2 - U(q)$

$$\frac{\partial L}{\partial \dot{q}} = a(q)\dot{q} \ , \quad \frac{\partial L}{\partial q} = \frac{1}{2}\frac{\partial a}{\partial q}\dot{q}^2 - \frac{\partial U}{\partial q}$$

L's eq:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \frac{d}{dt}(a(q)\dot{q}) - \frac{1}{2}\frac{da}{dq}\dot{q}^2 + \frac{dU}{dq} =$$

$$a(q)\ddot{q} + \frac{da}{dq}\dot{q}^2 - \frac{1}{2}\frac{da}{dq}\dot{q}^2 + \frac{dU}{dq} =$$

$$a(q)\ddot{q} + \frac{1}{2}\frac{da}{dq}\dot{q}^2 + \frac{dU}{dq} = 0$$

c) $E = \frac{1}{2}a(q)\dot{q}^2 + U(q) \Rightarrow \dot{E} = \frac{1}{2}\frac{da}{dq}\dot{q}^3 +$

$$\frac{1}{2}a(q)2\dot{q}\ddot{q} + \frac{dU}{dq}\dot{q} = \left(a(q)\ddot{q} + \frac{1}{2}\frac{da}{dq}\dot{q}^2 + \frac{dU}{dq}\right)\dot{q} = 0$$

according to L's eq.

$$E = \frac{1}{2}a(q)\dot{q}^2 + U(q) \Rightarrow \dot{q} = \sqrt{\frac{2(E - U(q))}{a(q)}} \Rightarrow$$

$$dt = \frac{\sqrt{a(q)}}{\sqrt{2(E-U(q))}}\,dq \qquad \text{if} \quad U(q) < E = \frac{mv_0^2}{2} + U(q_0).$$

This implies:

$$t = \int_0^t dt = \int_{q_0}^{q(t)} \frac{\sqrt{a(x)}}{\sqrt{2(E-U(x))}}\,dx\ ,\quad E = \frac{m v_0^2}{2} + U(q_0)$$

$a(q) = m\ ,\quad U(q) = \frac{k}{2} q^2$ (elastic potential)

$\Rightarrow$

$$t = \int_{q_0}^{q} \frac{\sqrt{m}}{\sqrt{2(E-\frac{1}{2}kx^2)}}\,dx = \sqrt{\frac{m}{k}} \int_{q_0}^{q} \frac{dx}{\sqrt{\frac{2E}{k} - x^2}} =$$

$$\left\{ \begin{array}{l} x = \sqrt{\frac{2E}{k}} \sin\varphi \\ dx = \sqrt{\frac{2E}{k}} \cos\varphi\, d\varphi \end{array} \right\} = \sqrt{\frac{m}{k}} \int_{\varphi_0}^{\varphi} d\varphi \quad \Rightarrow$$

$$t = \sqrt{\frac{m}{k}}(\varphi - \varphi_0) \Rightarrow \varphi = \omega_0 t + \varphi_0\ ,\quad \omega_0 = \sqrt{\frac{k}{m}}$$

The motion: $q = \sqrt{\frac{2E}{k}} \sin(\omega_0 t + \varphi_0)$

where $\varphi_0 = \arcsin\left(q_0\sqrt{\frac{k}{2E}}\right)$.

d) $q = q_0\ ,\ \dot q = \ddot q = 0$ is a solution to L's eq:

$$a(q)\ddot q + \frac{1}{2}\frac{da}{dq}\dot q^2 + \frac{dU}{dq} = 0$$

if and only if

$$\frac{dU}{dq}(q_0) = 0$$

$\frac{d^2U}{dq^2}(q_0) > 0 \Rightarrow U = U(q)$ has a strict local minimum at $q = q_0 \Rightarrow$ stable equilibrium according to Dirichlet.

#

3. L's eq. $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} = Q_k \Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) = 0 \Rightarrow \frac{\partial L}{\partial \dot q_k} = \text{const}.$$

#

4. $L^*(t, q^*, \dot q^*) = L(t, q, \dot q)$

$$E^* = \sum_{i=1}^n \frac{\partial L^*}{\partial \dot q_i^*} \dot q_i^* - L^*,\quad E = \sum_{i=1}^n \frac{\partial L}{\partial \dot q_i} \dot q_i - L$$

$$q_i = q_i(t, q_1^*, \dots, q_n^*) \Rightarrow \dot q_i = \frac{\partial q_i}{\partial t} + \sum_{j=1}^n \frac{\partial q_i}{\partial q_j^*} \dot q_j^* \Rightarrow$$

$$\frac{\partial \dot q_i}{\partial \dot q_j^*} = \frac{\partial q_i}{\partial q_j^*}.$$

$$E^* = \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial L}{\partial \dot q_j} \frac{\partial q_j}{\partial q_i^*}\right) \dot q_i^* - L^* =$$

$$\sum_{j=1}^n \frac{\partial L}{\partial \dot q_j} \sum_{i=1}^n \frac{\partial q_j}{\partial q_i^*} \dot q_i^* + E - \sum_{i=1}^n \frac{\partial L}{\partial \dot q_i} \dot q_i =$$

$$\sum_{j=1}^n \frac{\partial L}{\partial \dot q_j}\left(\dot q_j - \frac{\partial q_j}{\partial t}\right) + E - \sum_{i=1}^n \frac{\partial L}{\partial \dot q_i} \dot q_i = E$$

$$-\sum_{j=1}^n \frac{\partial L}{\partial \dot q_j} \frac{\partial q_j}{\partial t}$$

Q.E.D.

#

5. We have

$$b_{ij} = \left(\frac{\partial^2 L}{\partial \dot q_i \partial q_j}\right)_0 - \left(\frac{\partial^2 L}{\partial \dot q_j \partial q_i}\right)_0 - \left(\frac{\partial Q_i^{int}}{\partial q_j}\right)_0$$

$$Q_i^{int} = -\frac{\partial D}{\partial \dot q_i} + K_i^g = -\frac{\partial D}{\partial \dot q_i} + \sum_{j=1}^{n} \tilde g_{ij} \dot q_j$$

$$\frac{\partial Q_i^{int}}{\partial \dot q_j} = -\frac{\partial^2 D}{\partial \dot q_j \partial \dot q_i} + \tilde g_{ij} \quad \Rightarrow$$

$$b_{ij} = \underbrace{\left(\frac{\partial^2 D}{\partial \dot q_j \partial \dot q_i}\right)_0}_{= c_{ij}} + \underbrace{\left(\frac{\partial^2 L}{\partial \dot q_i \partial q_j}\right)_0 - \left(\frac{\partial^2 L}{\partial \dot q_j \partial q_i}\right)_0 - (\tilde g_{ij})_0}_{= g_{ij}}$$

since the first term is symmetric and the second anti-symmetric. Q.E.D

\#

6. a) $T = T(\dot\theta) = \frac{1}{2} J \dot\theta^2$, $V = V(\theta) = -mgh\cos\theta$

$Q^{ext} = 0$; $(L = T - V)$ L's eq.:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = Q^{ext} \Rightarrow \frac{d}{dt}(J\dot\theta) + mgh\sin\theta = 0$$

$$\underline{J\ddot\theta + mgh\sin\theta = 0}$$

b) $Q^{ext} = -M_f\, \mathrm{sign}(\dot\theta)$, $\mathrm{sign}(\dot\theta) = \begin{cases} 1 & \dot\theta > 0 \\ 0 & \dot\theta = 0 \\ -1 & \dot\theta < 0 \end{cases}$

$$J\ddot\theta + mgh\sin\theta = -M_f\, \mathrm{sign}(\dot\theta)$$

Equilibrium $\Rightarrow \frac{dV}{d\theta} = 0 \Rightarrow mgh\sin\theta = 0$

$\Rightarrow \theta = \theta_0 = 0, \pi$. (equilibrium conf.)

$\theta_0 = 0 \Rightarrow \frac{d^2V}{d\theta^2}(\theta_0) = mgh\cos\theta_0 = mgh > 0 \Rightarrow$

stable equilibrium.

$\theta_0 = \pi \Rightarrow \frac{d^2V}{d\theta^2}(\theta_0) = mgh\cos\theta_0 = -mgh < 0$

$\Rightarrow$ unstable equilibrium.

\#

7. Generalized coordinates: x, φ

$T = T(\varphi, \dot x, \dot\varphi) = \frac{1}{2} m_c \dot x^2 + \frac{1}{2} m\{(\dot x + l\cos\varphi\,\dot\varphi)^2 + (l\sin\varphi\,\dot\varphi)^2\} = \frac{1}{2}(m_c + m)\dot x^2 + \frac{1}{2} m l^2 \dot\varphi^2 + ml\cos\varphi\,\dot x\dot\varphi$

$V = V(x, \varphi) = \frac{1}{2} k x^2 - mgl\cos\varphi$

Lagrangian: $L = T - V$

$\frac{\partial L}{\partial \dot x} = (m_c + m)\dot x + ml\cos\varphi\,\dot\varphi$, $\frac{\partial L}{\partial \dot\varphi} = ml^2\dot\varphi + ml\cos\varphi\,\dot x$, $\frac{\partial L}{\partial x} = -kx$, $\frac{\partial L}{\partial \varphi} = -ml\sin\varphi\,\dot x\dot\varphi - mgl\sin\varphi$

Generalized forces: $Q_x = F$, $Q_\varphi = 0$

L's eq.:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot x}\right) - \frac{\partial L}{\partial x} = Q_x \Rightarrow \underline{(m_c+m)\ddot x +}$$

$$\underline{ml\cos\varphi\,\ddot\varphi - ml\sin\varphi\,\dot\varphi^2 + kx = F}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = Q_\varphi \Rightarrow ml^2\ddot\varphi + ml\cos\varphi\,\ddot x$$

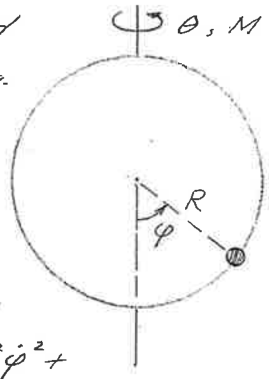
$$- ml\sin\varphi\,\dot x\dot\varphi + ml\sin\varphi\,\dot x\dot\varphi + mgl\sin\varphi = 0$$

or

$$\underline{ml^2\ddot\varphi + ml\cos\varphi\,\ddot x + mgl\sin\varphi = 0}$$

#

8. a) Introduce generalized coord. θ, φ according to fig.



$T = T(\dot\theta, \dot\varphi) = \frac{1}{2}J\dot\theta^2 +$

$\frac{1}{2}m\left((R\dot\varphi)^2 + (R\sin\varphi\,\dot\theta)^2\right)$.

Hoop inertia: $J = \frac{1}{2}m_h R^2$

$T = T(\dot\theta, \dot\varphi) = \frac{1}{4}m_h R^2\dot\theta^2 + \frac{1}{2}mR^2\dot\varphi^2 +$

$\frac{1}{2}mR^2\sin^2\varphi\,\dot\theta^2$, $V = V(\varphi) = -mgR\cos\varphi$

Lagrangean: $L = L(\varphi, \dot\theta, \dot\varphi) = \frac{1}{4}m_h R^2\dot\theta^2 +$

$\frac{1}{2}mR^2\dot\varphi^2 + \frac{1}{2}mR^2\sin^2\varphi\,\dot\theta^2 + mgR\cos\varphi$

Gen. forces: $Q_\theta = M = M(t)$, $Q_\varphi = 0$

L's eq.:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = \underline{\frac{1}{2}m_h R^2\ddot\theta + mR^2\sin^2\varphi\,\ddot\theta +}$$

$$\underline{mR^2\, 2\sin\varphi\cos\varphi\,\dot\varphi\dot\theta = M}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = \underline{mR^2\ddot\varphi - mR^2\sin\varphi\cos\varphi\,\dot\theta^2 +}$$

$$\underline{mgR\sin\varphi = 0}$$

b) $\frac{\partial L}{\partial \theta} \equiv 0 \Rightarrow \frac{\partial L}{\partial \dot\theta} = \text{const.} \Leftrightarrow M = 0$

$\frac{\partial L}{\partial \varphi} \neq 0 \Rightarrow \frac{\partial L}{\partial \dot\varphi}$ no integral of motion

$\frac{\partial L}{\partial t} \equiv 0 \Rightarrow \dot E = M\dot\theta = 0 \Leftrightarrow M = 0$

No integrals of motion were found.

c) $\dot\theta = \Omega = \text{const.} \Rightarrow$

$(M = 2mR^2\Omega\sin\varphi\cos\varphi\,\dot\varphi)$

$mR^2\ddot\varphi - mR^2\sin\varphi\cos\varphi\,\Omega^2 + mgR\sin\varphi = 0$

Equilibrium (relative): $\varphi = \varphi_0$ $\dot\varphi = \ddot\varphi = 0$

$\Rightarrow -mR^2\sin\varphi_0\cos\varphi_0\,\Omega^2 + mgR\sin\varphi_0 = 0$

$mR\sin\varphi_0(\Omega^2 R\cos\varphi_0 - g) = 0 \Rightarrow$

$$\varphi_0 = 0, \pi$$

$$\varphi_0 = \arccos\left(\frac{g}{\Omega^2 R}\right) \quad \text{if } \Omega > \sqrt{\frac{g}{R}}$$

But in this case:

$$T = \underbrace{\tfrac{1}{4} m_h R^2 \Omega^2 + \tfrac{1}{2} m R^2 \Omega^2 \sin^2\varphi}_{= T_0} + \underbrace{\tfrac{1}{2} m R^2 \dot\varphi^2}_{T_2}$$

$$V^* = V - T_0 = -mgR\cos\varphi - \tfrac{1}{4} m_h R^2 \Omega^2 -$$

$$\tfrac{1}{2} m R^2 \Omega^2 \sin^2\varphi$$

$$\frac{dV^*}{d\varphi}(\varphi_0) = mgR\sin\varphi_0 - mR^2\Omega^2 \sin\varphi_0 \cos\varphi_0 = 0$$

$$\frac{d^2V^*}{d\varphi^2}(\varphi_0) = mgR\cos\varphi_0 - mR^2\Omega^2\cos 2\varphi_0$$

$\underline{\varphi_0 = 0}$ $\Rightarrow$ $\dfrac{d^2V^*}{d\varphi^2} = mgR - mR^2\Omega^2 = mR(g - R\Omega^2)$

thus

$\Omega < \sqrt{\frac{g}{R}} \Rightarrow \varphi = 0$ stable equilibrium

$\Omega > \sqrt{\frac{g}{R}} \Rightarrow \varphi = 0$ unstable equilibrium

$\underline{\varphi_0 = \pi}$ $\Rightarrow$ $\dfrac{d^2V^*}{d\varphi^2} = -mgR - mR^2\Omega^2 < 0 \Rightarrow$

unstable equilibrium. Now

$$\varphi_0 = \arccos\left(\frac{g}{\Omega^2 R}\right) \Rightarrow \frac{d^2V^*}{d\varphi^2} = mgR\frac{g}{\Omega^2 R} -$$

$$mR^2\Omega^2\left(2\frac{g^2}{\Omega^4 R^2} - 1\right) = \frac{mg^2}{\Omega^2} - \frac{2mg^2}{\Omega^2} + mR^2\Omega^2$$

$$= -\frac{mg^2}{\Omega^2} + mR^2\Omega^2 = mR^2\Omega^2\left(1 - \frac{g^2}{R^2\Omega^4}\right) > 0$$

if $\Omega > \sqrt{\frac{g}{R}}$.

Thus:

$\Omega > \sqrt{\frac{g}{R}} \Rightarrow \varphi = \arccos\left(\frac{g}{\Omega^2 R}\right)$ stable equilibrium.

\#

9. a) Let (x_1, y_1) and (x_2, y_2) be cartesian coordinates for P_1 and P_2 respectively. We have: (θ, φ generalized coordinates)

$$x_1 = l_1\sin\theta, \quad y_1 = -l_1\cos\theta$$

$$x_2 = l_1\sin\theta + l_2\sin\varphi, \quad y_2 = -l_1\cos\theta - l_2\cos\varphi$$

$\Rightarrow$

$$\dot x_1 = l_1\cos\theta\,\dot\theta, \quad \dot y_1 = l_1\sin\theta\,\dot\theta$$

$$\dot x_2 = l_1\cos\theta\,\dot\theta + l_2\cos\varphi\,\dot\varphi, \quad \dot y_2 = l_1\sin\theta\,\dot\theta + l_2\sin\varphi\,\dot\varphi$$

$$T = T(\theta,\varphi,\dot\theta,\dot\varphi) = \tfrac{1}{2}m_1(\dot x_1^2+\dot y_1^2) + \tfrac{1}{2}m_2(\dot x_2^2+\dot y_2^2) =$$

$$\tfrac{a}{2}\dot\theta^2 + \tfrac{b}{2}\dot\varphi^2 + c\cos(\theta-\varphi)\dot\theta\dot\varphi \quad \text{where}$$

$$a = (m_1+m_2)l_1^2, \quad b = m_2 l_2^2, \quad c = m_2 l_1 l_2$$

$$V = V(\theta,\varphi) = m_1 g y_1 + m_2 g y_2 = -d\cos\theta - e\cos\varphi,$$

where $d = (m_1+m_2)g l_1$, $e = m_2 g l_2$

Lagrangian: $L = T - V = \tfrac{a}{2}\dot\theta^2 + \tfrac{b}{2}\dot\varphi^2 + c\cos(\theta-\varphi)\dot\theta\dot\varphi + d\cos\theta + e\cos\varphi$

$$\frac{\partial L}{\partial\dot\theta} = \frac{\partial T}{\partial\dot\theta} = a\dot\theta + c\cos(\theta-\varphi)\dot\varphi$$

$$\frac{\partial L}{\partial\dot\varphi} = \frac{\partial T}{\partial\dot\varphi} = b\dot\varphi + c\cos(\theta-\varphi)\dot\theta$$

$$\frac{\partial L}{\partial\theta} = \frac{\partial T}{\partial\theta} - \frac{\partial V}{\partial\theta} = -c\sin(\theta-\varphi)\dot\theta\dot\varphi - d\sin\theta$$

$$\frac{\partial L}{\partial\varphi} = \frac{\partial T}{\partial\varphi} - \frac{\partial V}{\partial\varphi} = c\sin(\theta-\varphi)\dot\theta\dot\varphi - e\sin\varphi$$

<u>L's eq.</u>:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} = a\ddot\theta + c\cos(\theta-\varphi)\ddot\varphi + c\sin(\theta-\varphi)\dot\varphi^2 + d\sin\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi}\right) - \frac{\partial L}{\partial\varphi} = b\ddot\varphi + c\cos(\theta-\varphi)\ddot\theta - c\sin(\theta-\varphi)\dot\theta^2 + e\sin\varphi = 0$$

b) Linearization $\Rightarrow$ $\cos\theta \cong \cos\varphi \cong 1$, $\sin\theta \cong \theta$, $\sin\varphi \cong \varphi$ and then

$$\begin{cases} a\ddot\theta + c\ddot\varphi + d\theta = 0 \\ c\ddot\theta + b\ddot\varphi + e\varphi = 0 \end{cases}$$

or in matrix-form:

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} \ddot\theta \\ \ddot\varphi \end{pmatrix} + \begin{pmatrix} d & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

\#

See separate solution

10. a) $T = \tfrac{1}{2}m_2\dot x^2 + \tfrac{1}{2}m_1\{(\dot x - l\sin\theta\dot\theta)^2 + (l\cos\theta\dot\theta)^2\} = \tfrac{1}{2}m\dot x^2 + \tfrac{1}{2}m_1 l^2\dot\theta^2 - m_1 l\sin\theta\,\dot x\dot\theta$, where $m = m_1 + m_2$

$$V = -mg(x + l\cos\theta) + \tfrac{1}{2}k_1x^2 + \tfrac{1}{4}k_2x^4$$

$$Q_x = -F_f\,\mathrm{sign}(\dot x), \quad Q_\theta = 0$$

Lagrangian: $L = T - V = \tfrac{1}{2}m\dot x^2 + \tfrac{1}{2}m_1 l^2\dot\theta^2 - m_1 l\sin\theta\,\dot x\dot\theta + mg(x + l\cos\theta) - \tfrac{1}{2}k_1x^2 - \tfrac{1}{4}k_2x^4$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} - m_1 l \sin\theta\,\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 l^2 \dot{\theta} - m_1 l \sin\theta\,\dot{x}$$

$$\frac{\partial L}{\partial x} = mg - k_1 x - k_2 x^3$$

$$\frac{\partial L}{\partial \theta} = - m_1 l \cos\theta\,\dot{x}\dot{\theta} - m_1 g l \sin\theta$$

L's eq.:

$$(m_1 + m_2)\ddot{x} - m_1 l \sin\theta\,\ddot{\theta} - m_1 l \cos\theta\,\dot{\theta}^2 - mg + k_1 x + k_2 x^3 = - F_f\,\mathrm{sign}(\dot{x})$$

$$m_1 l^2 \ddot{\theta} - m_1 l \sin\theta\,\ddot{x} + m_1 g l \sin\theta = 0$$

b) Dissipation function: $D = D(x, \theta, \dot{x}, \dot{\theta})$

$$\left.\begin{aligned} Q_x &= -F_f\,\mathrm{sign}(\dot{x}) = -\frac{\partial D}{\partial \dot{x}} \\ Q_\theta &= 0 = -\frac{\partial D}{\partial \dot{\theta}} \end{aligned}\right\} \Rightarrow D = F_f |\dot{x}| + \underline{\underline{G(x,\theta)}}$$

c) Equilibrium ($x = x_0$, $\theta = \theta_0$, $\dot{x} = \dot{\theta} = 0$, $\ddot{x} = \ddot{\theta} = 0$) $\Rightarrow$

$$-mg + k_1 x_0 + k_2 x_0^3 = 0 \quad \Rightarrow$$

$$x_0 = \left(\frac{mg}{2k_2} + \sqrt{\left(\frac{k_1}{3k_2}\right)^3 + \left(\frac{mg}{2k_2}\right)^2}\right)^{\frac{1}{3}} + \left(\frac{mg}{2k_2} - \sqrt{\left(\frac{k_1}{3k_2}\right)^3 + \left(\frac{mg}{2k_2}\right)^2}\right)^{\frac{1}{3}} > 0$$

$$mgl \sin\theta_0 = 0 \quad \Rightarrow \quad \theta_0 = 0, \pi$$

$$\frac{\partial V}{\partial x} = -mg + k_1 x + k_2 x^3$$

$$\frac{\partial V}{\partial \theta} = m_1 g l \sin\theta$$

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_0 = k_1 + 3k_2 x_0^2, \quad \left(\frac{\partial^2 V}{\partial \theta^2}\right)_0 = m_1 g l \cos\theta_0$$

$$\left(\frac{\partial^2 V}{\partial \theta \partial x}\right)_0 = \left(\frac{\partial^2 V}{\partial x \partial \theta}\right)_0 = 0$$

Thus:

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_0 > 0, \quad \left(\frac{\partial^2 V}{\partial \theta^2}\right)_0 = \begin{cases} m_1 g l > 0 & \theta_0 = 0 \\ -m_1 g l < 0 & \theta_0 = \pi \end{cases}$$

and consequently:

$\theta_0 = 0$, $x = x_0$ stable equilibrium

$\theta_0 = \pi$, $x = x_0$ unstable equilibrium

The energy balance $\Rightarrow$

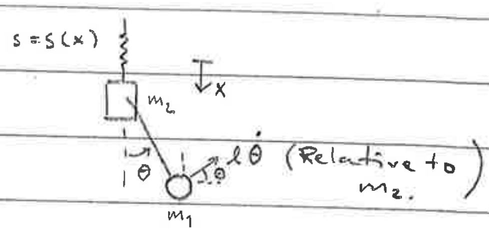
$$\dot{E} = \dot{T} + \dot{V} = Q_x \dot{x} = - F_f \, \mathrm{sign}(\dot{x}) \dot{x} =$$

$$-F_f |\dot{x}| < 0 \quad , \quad \dot{x} \neq 0$$ $\Rightarrow$ the equilibrium $x = x_0$, $\theta_0 = 0$ is asymptotically stable.

#

1:10

a)



• Kinetic energy: {Page 40}

$$T = T(\theta, \dot{\theta}, \dot{x}) = \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_1 \left[(\dot{x} - l\dot{\theta}\sin\theta)^2 + (l\cos\theta\dot{\theta})^2 \right] =$$

$$= \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_1 \left[\dot{x}^2 - 2\dot{x} l \dot{\theta} \sin\theta + l^2 \dot{\theta}^2 \sin^2\theta + l^2 \dot{\theta}^2 \cos^2\theta \right] =$$

$$= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_1 l^2 \dot{\theta}^2 - m_1 l \sin\theta \dot{x} \dot{\theta}$$

• Potential energy: {Pages 25 and 49}

$$V = V(x, \theta) = -m_1 g(x + l\cos\theta) - m_2 g x +$$

$$+ \frac{1}{2} k_1 x^2 + \frac{1}{4} k_2 x^4$$

• Generalized forces:

$$Q_x = -F_f \, \mathrm{sign}(\dot{x}), \quad \mathrm{sign}(\dot{x}) = \begin{cases} 1 & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ -1 & \dot{x} < 0 \end{cases}$$

$$Q_\theta = 0$$

• Lagrangian function: {Page 47 (6.7)}

$$L = T - V = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_1 l^2 \dot{\theta}^2 - m_1 l \sin\theta \dot{x} \dot{\theta} +$$

$$+ m_1 g(x + l\cos\theta) + m_2 g x - \frac{1}{2} k_1 x^2 - \frac{1}{4} k_2 x^4$$

1 : 10

a)

$$\frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} - m_1 l \sin\theta\,\dot{\theta}$$

$$\frac{\partial L}{\partial \dot{\theta}} = m_1 l^2 \dot{\theta} - m_1 l \sin\theta\,\dot{x}$$

$$\frac{\partial L}{\partial x} = m_1 g + m_2 g - k_1 x - k_2 x^3$$

$$\frac{\partial L}{\partial \theta} = - m_1 l \cos\theta\,\dot{x}\,\dot{\theta} - m_1 g l \sin\theta$$

- Lagranges equation: {Page 71 (11.1)}

$$(*)\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = (m_1 + m_2)\ddot{x} - m_1 l\left(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}\right) +$$

$$- m_1 g - m_2 g + k_1 x + k_2 x^3 =$$

$$= Q_x = - F_f \operatorname{sign}(\dot{x})$$

$$(**)\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m_1 l^2 \ddot{\theta} - m_1 l\left(\cos\theta\,\dot{\theta}\,\dot{x} + \sin\theta\,\ddot{x}\right) +$$

$$+ m_1 l \cos\theta\,\dot{x}\,\dot{\theta} + m_1 g l \sin\theta =$$

$$= m_1 l^2 \ddot{\theta} - m_1 l \sin\theta\,\ddot{x} + m_1 g l \sin\theta = Q_\theta = 0$$

1:10

b)

- Dissipation function: {Page 54 (7.3)}

$$D = D(x, \theta, \dot{x}, \dot{\theta})$$

$$\left.\begin{aligned} Q_x &= -F_f\, \mathrm{sign}(\dot{x}) = -\frac{\partial D}{\partial \dot{x}} \\ Q_\theta &= 0 = -\frac{\partial D}{\partial \dot{\theta}} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow D = F_f |\dot{x}| + \underbrace{G(x, \theta)}$$

Does not influence the dynamics of the system and could be chosen as $G(x, \theta) = 0$.

1 : 10

c)

- Equilibrium : ⊛

$$x = x_0 \ , \quad \theta = \theta_0 \ , \quad \dot{x} = \dot{\theta} = 0 \ , \quad \ddot{x} = \ddot{\theta} = 0$$

$$(*) \quad -m_1 g - m_2 g + k_1 x_0 + k_2 x_0^3 = 0 \quad \Rightarrow$$

$$x_0 = \left(\frac{(m_1+m_2)g}{2k_2} + \sqrt{\left(\frac{k_1}{3k_2}\right)^3 + \left[\frac{(m_1+m_2)g}{2k_2}\right]^2} \right)^{\frac{1}{3}} +$$

$$+ \left(\frac{(m_1+m_2)g}{2k_2} - \sqrt{\left(\frac{k_1}{3k_2}\right)^3 + \left[\frac{(m_1+m_2)g}{2k_2}\right]^2} \right)^{\frac{1}{3}} > 0$$

$$(**) \quad m_1 g l \sin\theta_0 = 0 \quad \Rightarrow \quad \theta_0 = 0, \pi$$

$$\frac{\partial V}{\partial x} = -m_1 g - m_2 g + k_1 x + k_2 x^3$$

$$\frac{\partial V}{\partial \theta} = m_1 g l \sin\theta$$

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_0 = k_1 + 3k_2 x_0^2$$

$$\left(\frac{\partial^2 V}{\partial \theta^2}\right)_0 = m_1 g l \cos\theta_0$$

$$\left(\frac{\partial^2 V}{\partial x \partial \theta}\right)_0 = \left(\frac{\partial^2 V}{\partial \theta \partial x}\right)_0 = 0$$

⊛ $\frac{\partial V^*}{\partial q_i}(q_i^0) = 0$, $V^* = V - T_0$, here $T_0 = 0$ {Pages 40 and 66}

7 : 10

c)

$$\left(\frac{\partial^2 V}{\partial x^2}\right)_0 > 0$$

$\theta_0 = 0$ $\quad \left(\frac{\partial^2 V}{\partial \theta^2}\right)_0 = m_1 g l > 0$

$\theta_0 = \pi$ $\quad \left(\frac{\partial^2 V}{\partial \theta^2}\right)_0 = - m_1 g l < 0$

$\theta_0 = 0$, $x = x_0$ stable equilibrium

$\theta_0 = \pi$, $x = x_0$ unstable equilibrium

MECHANICAL VIBRATIONS

Exercise 2: Undamped vibration.

Problem solutions

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1. $\bar{q} = \bar{q}(t) \bar{x} e^{st} \Rightarrow \dot{\bar{q}} = s\bar{x}e^{st} \Rightarrow \ddot{\bar{q}} = s^2 \bar{x} e^{st}$

$\Rightarrow \underline{M}\ddot{\bar{q}} + \underline{B}\dot{\bar{q}} + \underline{K}\bar{q} = (\underline{M}s^2 + \underline{B}s + \underline{K})\bar{x}e^{st} = \bar{0}$

$\forall t \Rightarrow (\underline{M}s^2 + \underline{B}s + \underline{K})\bar{x} = \underline{Z}(s)\bar{x} = \bar{0}$

--- #

2. $\bar{q} = \bar{u}(\alpha t + \beta) \Rightarrow \dot{\bar{q}} = \bar{u}\alpha \Rightarrow \ddot{\bar{q}} = \bar{0} \Rightarrow$

$\underline{M}\ddot{\bar{q}} + \underline{K}\bar{q} = \underline{K}\bar{u}(\alpha t + \beta)$. $\det \underline{K} = 0 \Rightarrow$

$\exists\, \bar{u} \neq \bar{0}$ such that $\underline{K}\bar{u} = \bar{0}$

--- #

3. In normalized coordinates we have:

$$R(\bar{x}) = \frac{\sum_{i=1}^{n} \omega_i^2 \xi_i^2}{\sum_{i=1}^{n} \xi_i^2}$$

Choose $\sum_{i=1}^{n} \xi_i^2 = 1 \Rightarrow \xi_1^2 = 1 - \sum_{i=2}^{n} \xi_i^2 \Rightarrow$

$R(\bar{x}) = \omega_1^2 + \sum_{i=2}^{n} (\omega_i^2 - \omega_1^2)\xi_i^2 \geqq \omega_1^2$, since $\omega_i^2 \geqq \omega_1^2$

On the other hand: $\xi_n^2 = 1 - \sum_{i=1}^{2} \xi_i^2 \Rightarrow$

$R(\bar{x}) = \omega_n^2 + \sum_{i=2}^{n} (\omega_i^2 - \omega_n^2)\xi_i^2 \leqq \omega_n^2$ since $\omega_i^2 \leqq \omega_n^2$

--- #

4. a)

$\det(-\omega^2 \underline{M} + \underline{K}) = c_n(\omega^2)^n + c_{n-1}(\omega^2)^{n-1} + \dots + c_0$

$\omega^2 = 0 \Rightarrow \det \underline{K} = c_0$

$\det(-\omega^2 \underline{M} + \underline{K}) = (-1)^n (\omega^2)^n \det(\underline{M} - \frac{1}{\omega^2}\underline{K}) =$

$c_n(\omega^2)^n + c_{n-1}(\omega^2)^{n-1} + \dots + c_0 \Rightarrow \det(\underline{M} - \frac{1}{\omega^2}\underline{K}) =$

$(-1)^n \{ c_n + c_{n-1}\frac{1}{\omega^2} + \dots + \frac{c_0}{(\omega^2)^n} \}$, $\omega^2 > 0$

Thus:

$\lim_{\omega^2 \to \infty} \det(\underline{M} - \frac{1}{\omega^2}\underline{K}) = \det\left(\lim_{\omega^2 \to \infty} (\underline{M} - \frac{1}{\omega^2}\underline{K})\right) =$

$\det \underline{M} = (-1)^n c_n \Rightarrow c_n = (-1)^n \det \underline{M}$

b) $\det \underline{K} = 0 \Rightarrow c_0 = 0 \Rightarrow \omega^2$ is a root.

c) Factorization gives:

$c_n(\omega^2)^n + c_{n-1}(\omega^2)^{n-1} + \dots + c_0 =$

$c_n(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \dots (\omega^2 - \omega_n^2)$

$\omega^2 = 0 \Rightarrow c_0 = c_n(-1)^n \omega_1^2 \omega_2^2 \dots \omega_n^2 \Rightarrow$

$\det \underline{K} = (-1)^n \det \underline{M} (-1)^n \omega_1^2 \omega_2^2 \dots \omega_n^2 \Rightarrow$

$\omega_1^2 \omega_2^2 \dots \omega_n^2 = \frac{\det \underline{K}}{\det \underline{M}}$.

--- #

5 a) From the solution of 1:7 we get the <u>linearized equations</u> of motion:

$$(m_c+m)\ddot{x} + ml\ddot{\varphi} + kx = 0$$
$$ml\ddot{x} + ml^2\ddot{\varphi} + mgl\varphi = 0$$

in <u>matrix form</u>:

$$\begin{pmatrix} m_c+m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} k & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} x \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{M}\ \ddot{\bar{q}} + \underline{K}\ \bar{q} = \bar{0}$$

b) <u>Eigenfrequencies</u>:

$$\det(-\omega^2\underline{M} + \underline{K}) = \begin{vmatrix} -\omega^2(m_c+m)+k & -\omega^2 ml \\ -\omega^2 ml & -\omega^2 ml^2 + mgl \end{vmatrix} =$$

$$= \omega^4 m_c m l^2 - \omega^2((m_c+m)mgl + kml^2) + kmgl = 0$$

$$\omega_{2,1}^2 = \frac{(m_c+m)g+kl}{2m_c l} \pm \sqrt{\left(\frac{(m_c+m)g+kl}{2m_c l}\right)^2 - \frac{kg}{m_c l}}$$

$m_c = m = 1kg$, $l = 1.0m$, $k = 100 N/m$

$g = 9.81 m/s^2$:

$$\omega_1^2 = 8.857\ s^{-2}, \quad \omega_2^2 = 110.763\ s^{-2}$$

<u>Mode shapes</u>:

$\underline{\omega_1^2 = 8.857\ s^{-2}}$

$$(-\omega_1^2\underline{M} + \underline{K})\bar{X}_1 = \bar{0} \quad \left(\bar{X}_1 = \begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix}\right) \Rightarrow$$

$$(-\omega_1^2(m_c+m)+k)X_{11} + \omega_1^2 ml X_{21} = 0 \Rightarrow$$

$$X_{21} = \frac{k-\omega_1^2(m_c+m)}{\omega_1^2 ml} X_{11} \Rightarrow$$

$$\bar{X}_1 = \begin{pmatrix} 1 \\ 9.291 \end{pmatrix}$$

$$\underline{\omega_2^2 = 110.763\ s^{-2}} \Rightarrow \underline{\bar{X}_2 = \begin{pmatrix} 1 \\ -1.097 \end{pmatrix}}$$

c) $$\bar{X}_1^T \underline{M}\,\bar{X}_2 = (1\quad 9.291)\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1.097 \end{pmatrix} =$$

$$(1\quad 9.291)\begin{pmatrix} 0.903 \\ -0.097 \end{pmatrix} = 0$$

d) The <u>admittance matrix</u>:

$$\underline{A}(i\omega) = \frac{\bar{X}_1\bar{X}_1^T}{(\omega_1^2-\omega^2)\mu_1} + \frac{\bar{X}_2\bar{X}_2^T}{(\omega_2^2-\omega^2)\mu_2}$$

$$\bar{x} = \underline{A}(i\omega)\,\bar{S}\sin\omega t, \quad \bar{S} = \begin{pmatrix} F_0 \\ 0 \end{pmatrix}$$

$$\bar{X}_1 \bar{X}_1^T = \begin{pmatrix} 1 & 9.291 \\ 9.291 & 86.32 \end{pmatrix} \qquad \mu_1 = \bar{X}_1^T \underline{M} \bar{X}_1 = 106.905$$

$$\bar{X}_2 \bar{X}_2^T = \begin{pmatrix} 1 & -1.097 \\ -1.097 & 1.203 \end{pmatrix} \qquad \mu_2 = \bar{X}_2^T \underline{M} \bar{X}_2 = 1.009$$

$$\bar{X} = \underline{A}(i\omega) \begin{pmatrix} F_0 \\ 0 \end{pmatrix} \sin\omega t = \begin{pmatrix} A(i\omega)_{11} \\ A(i\omega)_{21} \end{pmatrix} F_0 \sin\omega t$$

$$= \begin{pmatrix} \dfrac{0.0094}{8.857-\omega^2} + \dfrac{0.9911}{110.763-\omega^2} \\[2ex] \dfrac{0.086}{8.857-\omega^2} + \dfrac{-1.0872}{110.763-\omega^2} \end{pmatrix} F_0 \sin\omega t$$

\#

6. a) From the solution of Exercise 1:9 and with $m_1 = m_2 = m$, $l_1 = l_2 = l$. $\Rightarrow$

$$a = (m_1+m_2)l_1^2 = 2ml^2, \quad b = m_2 l_2^2 = ml^2$$

$$c = m_2 l_1 l_2 = ml^2, \quad d = (m_1+m_2)gl_1 = 2mgl$$

$$e = m_2 g l_2 = mgl$$

$$\begin{cases} 2ml^2\ddot\theta + ml^2\ddot\varphi + 2mgl\theta = 0 \\ ml^2\ddot\theta + ml^2\ddot\varphi + mgl\varphi = 0 \end{cases}$$

or in matrix form:

$$\begin{pmatrix} 2ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix}\begin{pmatrix} \ddot\theta \\ \ddot\varphi \end{pmatrix} + \begin{pmatrix} 2mgl & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{M} \quad \ddot{\bar{q}} \quad + \quad \underline{K} \quad \bar{q} = \bar{0}$$

b) Eigenfrequencies:

$$\det(-\omega^2\underline{M} + \underline{K}) = \begin{vmatrix} -\omega^2 2ml^2 + 2mgl & -ml^2\omega^2 \\ -ml^2\omega^2 & -\omega^2 ml^2 + mgl \end{vmatrix}$$

$$= (-\omega^2 2ml^2 + 2mgl)(-\omega^2 ml^2 + mgl) - \omega^4 m^2 l^4 =$$

$$= \omega^4 m^2 l^4 - \omega^2 4m^2 g l^3 + 2m^2 g^2 l^2 = 0$$

$$\omega_{2,1}^2 = \frac{2g}{l} \pm \sqrt{\frac{4g^2}{l^2} - 2\frac{g^2}{l^2}} = (2 \pm \sqrt{2})\frac{g}{l}$$

Mode shapes:

$\omega_1^2 = (2-\sqrt{2})\dfrac{g}{l}$: $\quad (-\omega_1^2\underline{M} + \underline{K})\bar{X}_1 = 0 \Rightarrow$

$$(-\omega_1^2 2ml^2 + 2mgl)X_{11} - ml^2\omega_1^2 X_{21} = 0 \Rightarrow$$

$$\frac{X_{21}}{X_{11}} = \frac{-\omega_1^2 2ml^2 + 2mgl}{ml^2\omega_1^2} = \frac{-(2-\sqrt{2})\frac{g}{l}2ml^2 + 2mgl}{ml^2(2-\sqrt{2})\frac{g}{l}}$$

$$= \frac{-(2-\sqrt{2})2+2}{2-\sqrt{2}} = \frac{2(\sqrt{2}-1)}{\sqrt{2}(\sqrt{2}-1)} = \sqrt{2}$$

$$\bar{x}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

$\underline{\omega_2^2 = (2+\sqrt{2})\frac{g}{l}}$: $\quad (-\omega_2^2 \underline{M} + \underline{K})\bar{x}_2 = \bar{0} \quad \Rightarrow$

$$(-\omega_2^2 2ml^2 + 2mgl)x_{12} - ml^2\omega_2^2 x_{22} = 0 \quad \Rightarrow$$

$$\frac{x_{22}}{x_{12}} = \frac{-\omega_2^2 2ml^2 + 2mgl}{ml^2\omega_2^2} = -\sqrt{2}$$

$$\bar{x}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

#

7.a) With generalized coordinates θ_1, θ_2:

$$T = T(\dot{\theta}_1, \dot{\theta}_2) = \frac{m}{2}\left((l\dot{\theta}_1)^2 + (l\dot{\theta}_2)^2\right) =$$

$$\frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) = \frac{1}{4}ml^2\{(\dot{\theta}_1+\dot{\theta}_2)^2 + (\dot{\theta}_1-\dot{\theta}_2)^2\}$$

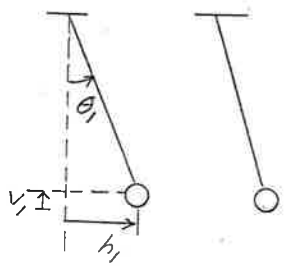
$$V = V(\theta_1, \theta_2) = mg(v_1+v_2) +$$

$$\frac{1}{2}k\left(\frac{h_2-h_1}{2}\right)^2 =$$

$$\frac{mgl}{2}(\theta_1^2+\theta_2^2) + \frac{1}{8}kl^2(\theta_2-\theta_1)^2$$

$$= \frac{mgl}{4}(\theta_1+\theta_2)^2 +$$

$$\frac{1}{4}(mg + \frac{1}{2}kl)l(\theta_1-\theta_2)^2$$



$h_1 = l\sin\theta_1 \simeq l\theta_1$

$v_1 = l(1-\cos\theta_1) \simeq \frac{l\theta_1^2}{2}$

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = ml^2\dot{\theta}_1, \quad \frac{\partial L}{\partial \theta_1} = -mgl\theta_1 - \frac{1}{4}kl^2(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = ml^2\dot{\theta}_2, \quad \frac{\partial L}{\partial \theta_2} = -mgl\theta_2 - \frac{1}{4}kl^2(\theta_2-\theta_1)$$

<u>Equations of motion:</u>

$$ml^2\ddot{\theta}_1 + (mgl + \frac{1}{4}kl^2)\theta_1 - \frac{1}{4}kl^2\theta_2 = 0$$

$$ml^2\ddot{\theta}_2 - \frac{1}{4}kl^2\theta_1 + (mgl + \frac{1}{4}kl^2)\theta_2 = 0$$

or

$$\underbrace{\begin{pmatrix} ml^2 & 0 \\ 0 & ml^2 \end{pmatrix}}_{\underline{M}} \underbrace{\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix}}_{\ddot{\bar{q}}} + \underbrace{\begin{pmatrix} mgl + \frac{1}{4}kl^2 & -\frac{1}{4}kl^2 \\ -\frac{1}{4}kl^2 & mgl + \frac{1}{4}kl^2 \end{pmatrix}}_{\underline{K}} \underbrace{\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}}_{\bar{q}} = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\bar{0}}$$

b) Introduce coordinates (normal coord.):

$$\eta_1 = l\sqrt{\frac{m}{2}}(\theta_1+\theta_2), \quad \eta_2 = l\sqrt{\frac{m}{2}}(\theta_1-\theta_2) \quad \Rightarrow$$

$$T = \frac{1}{2}(\dot{\eta}_1^2 + \dot{\eta}_2^2), \quad V = \frac{1}{2}\left(\frac{g}{l}\eta_1^2 + \frac{g}{l}(1+\frac{kl}{2mg})\eta_2^2\right)$$

and then we may directly read off:

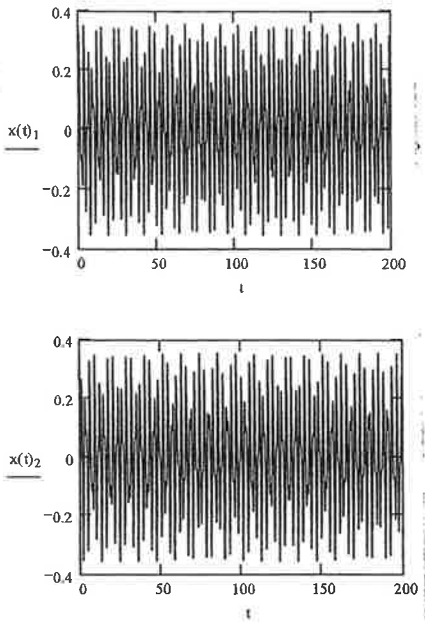
$$\omega_1^2 = \frac{g}{l}, \quad \bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\omega_2^2 = \frac{g}{l}\left(1 + \frac{kL}{2mg}\right), \quad \bar{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

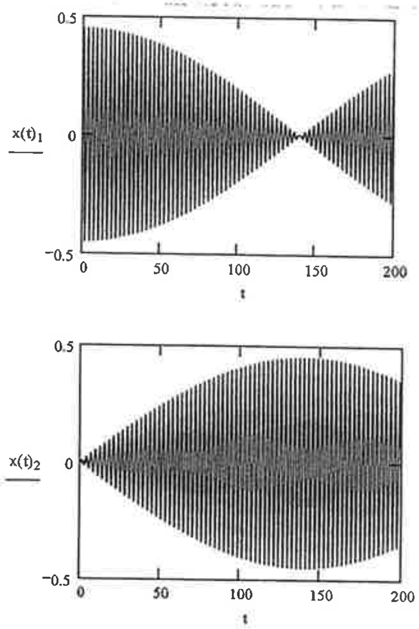
c) $\theta_1(0) = \theta_2(0) = 0$ $\qquad \bar{X}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\dot{\theta}_1(0) = 1\,rad/s, \quad \dot{\theta}_2(0) = 0$ $\qquad \dot{\bar{X}}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$k = 10\,N/m$:



$k = 0.1\,N/m$



"Beating"

The motion:

$$\bar{x}(t) = \left(\sum_{i=1}^{2} \frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i} \cos(\omega_i t)\right)\bar{x}_0 + \left(\sum_{i=1}^{2} \frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i \omega_i} \sin\omega_i t\right)\dot{\bar{x}}_0$$

$$\mu_i = \bar{x}_i^T \underline{M}\, \bar{x}_i$$

8. a) $T = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2)$

$V = \frac{k}{2}x_1^2 + \frac{k}{2}(x_2 - x_1)^2 + \frac{k}{2}x_2^2$

$Q_1 = F_0 \sin(\omega t), \quad Q_2 = 0$

$L = T - V$: $\quad \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1, \quad \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$

$\frac{\partial L}{\partial x_1} = -kx_1 - k(x_1 - x_2)$

$\frac{\partial L}{\partial x_2} = -k(x_2 - x_1) - kx_2$

Equations of motion

$$m\ddot{x}_1 + 2kx_1 - kx_2 = F_0 \sin\omega t$$

$$m\ddot{x}_2 - kx_1 + 2kx_2 = 0$$

b) $F_0 = 0$. Matrix form:

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\underline{M}\,\ddot{\bar{x}} + \underline{K}\,\bar{x} = \bar{0}$$

Eigenfrequencies:

$$\det(-\omega^2\underline{M} + \underline{K}) = \begin{vmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + 2k \end{vmatrix} =$$

$$(-\omega^2 m + 2k)^2 - k^2 = \omega^4 m^2 - \omega^2 4mk + 4k^2 - k^2 = 0$$

$$\Rightarrow \quad \omega^4 - \omega^2 \frac{4k}{m} + \frac{3k^2}{m^2} = 0 \quad \Rightarrow$$

$$\omega_{1,2}^2 = \frac{2k}{m} \pm \sqrt{\frac{4k^2}{m^2} - \frac{3k^2}{m^2}} = \frac{2k}{m} \pm \frac{k}{m} \Rightarrow$$

$$\omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{3k}{m}$$

Mode shapes

$$\underline{\omega_1^2 = \frac{k}{m}} : \quad (-\omega_1^2 \underline{M} + \underline{k})\bar{X}_1 = \bar{0} \quad \Rightarrow$$

$$\left(-\frac{k}{m}m + 2k\right)x_{11} - k x_{21} = \bar{0} \quad \Rightarrow$$

$$\frac{x_{21}}{x_{11}} = 1 \quad \Rightarrow \quad \bar{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{\omega_2^2 = \frac{3k}{m}} : \quad (-\omega_2^2 \underline{M} + \underline{k})\bar{X}_2 = \bar{0} \quad \Rightarrow$$

$$\left(-\frac{3k}{m}m + 2k\right)x_{12} - k x_{22} = 0 \quad \Rightarrow$$

$$\frac{x_{22}}{x_{12}} = -1 \quad \Rightarrow \quad \bar{X}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

c) $\bar{X} = \bar{X}(t) = \underline{A}(i\omega)\,\bar{S}\,\sin\omega t$

$$\bar{S} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} F_0$$

$\underline{A}(i\omega)$ admittance matrix

$$\underline{A}(i\omega) = \frac{\bar{X}_1 \bar{X}_1^T}{(\omega_1^2 - \omega^2)\mu_1} + \frac{\bar{X}_2 \bar{X}_2^T}{(\omega_2^2 - \omega^2)\mu_2}$$

$$\bar{X}_1 \bar{X}_1^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1 \quad 1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\bar{X}_2 \bar{X}_2^T = \begin{pmatrix} 1 \\ -1 \end{pmatrix}(1 \quad -1) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\mu_1 = \bar{X}_1^T \underline{M}\, \bar{X}_1 = (1 \quad 1)\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2m$$

$$\mu_2 = \bar{X}_2^T \underline{M}\, \bar{X}_2 = 2m$$

$$\underline{A}(i\omega) =$$

$$\begin{pmatrix} \frac{1}{(\frac{k}{m} - \omega^2)2m} + \frac{1}{(3\frac{k}{m} - \omega^2)2m} & \frac{1}{(\frac{k}{m} - \omega^2)2m} - \frac{1}{(3\frac{k}{m} - \omega^2)2m} \\ \frac{1}{(\frac{k}{m} - \omega^2)2m} - \frac{1}{(3\frac{k}{m} - \omega^2)2m} & \frac{1}{(\frac{k}{m} - \omega^2)2m} + \frac{1}{(3\frac{k}{m} - \omega^2)2m} \end{pmatrix}$$

$$\underline{A}(i\omega)\bar{S} = \begin{pmatrix} \frac{1}{(\frac{k}{m} - \omega^2)2m} + \frac{1}{(3\frac{k}{m} - \omega^2)2m} \\ \frac{1}{(\frac{k}{m} - \omega^2)2m} - \frac{1}{(3\frac{k}{m} - \omega^2)2m} \end{pmatrix} F_0$$

$$x_1(t) = \left(\frac{1}{\left(\frac{k}{m} - \omega^2\right)2m} + \frac{1}{\left(3\frac{k}{m} - \omega^2\right)2m} \right) F_0 \sin\omega t$$

$$= \frac{1}{2m} \frac{3\frac{k}{m} - \omega^2 + \frac{k}{m} - \omega^2}{\left(\frac{k}{m} - \omega^2\right)\left(\frac{3k}{m} - \omega^2\right)} F_0 \sin\omega t =$$

$$\frac{1}{m} \frac{\frac{2k}{m} - \omega^2}{\left(\frac{k}{m} - \omega^2\right)\left(\frac{3k}{m} - \omega^2\right)} F_0 \sin\omega t$$

$$x_1(t) \equiv 0 \iff \underline{\omega^2 = \frac{2k}{m}} \quad \text{(anti-resonance)}$$

#

9. a) $T = \frac{1}{2}\left(2m\dot{x}_1^2 + 3m\dot{x}_2^2 + m\dot{x}_3^2\right)$

$$= \frac{1}{2}\left(\dot{x}_1 \; \dot{x}_2 \; \dot{x}_3\right) \underbrace{\begin{pmatrix} 2m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & m \end{pmatrix}}_{\underline{M}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

$$V = \frac{1}{2}\left(2kx_1^2 + 3k(x_2 - x_1)^2 + 2k(x_3 - x_2)^2 + kx_3^2\right)$$

$$= \frac{1}{2}\left(x_1 \; x_2 \; x_3\right) \underbrace{\begin{pmatrix} 5k & -3k & 0 \\ -3k & 5k & -2k \\ 0 & -2k & 3k \end{pmatrix}}_{\underline{K}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Equations of motion: (free vibrations)

$$\underline{M}\ddot{\bar{x}} + \underline{K}\bar{x} = \bar{0}$$

b) Natural frequencies:

$\det(-\omega^2\underline{M} + \underline{K}) = 0 \implies$ (using for instance Mathcad)

$$\omega_1^2 = 0.426\frac{k}{m}, \quad \omega_2^2 = 2.741\frac{k}{m}, \quad \omega_3^2 = 4\frac{k}{m}$$

Mode shapes:

$$\omega_1^2 = 0.426\frac{k}{m}: \quad \bar{x}_1 = \begin{pmatrix} 1 \\ 1.383 \\ 1.074 \end{pmatrix} \qquad \begin{matrix} \omega_1^2 = 4260 \\ \omega_1 = 65.3 \end{matrix}$$

$\omega_2^2 = 2.741\frac{k}{m}$: $\quad \bar{X}_2 = \begin{pmatrix} 1 \\ -0.161 \\ -1.241 \end{pmatrix}$, $\omega_2^2 = 27410$

$\omega_2 = 165.5$

$\omega_3^2 = 4\frac{k}{m}$: $\quad \bar{X}_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\omega_3^2 = 40000$

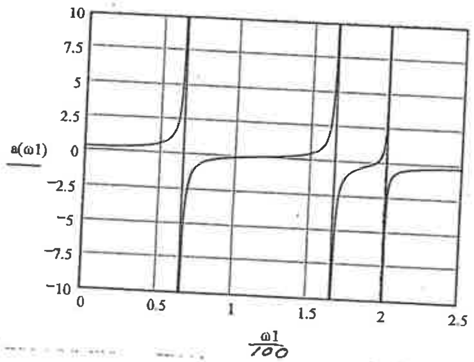
$\omega_3 = 200$

c) Modal matrix $\underline{X} = (\bar{X}_1 \; \bar{X}_2 \; \bar{X}_3)$

$$\underline{X}^T \underline{M} \underline{X} = \begin{pmatrix} 8.892 & 0 & 0 \\ 0 & 3.618 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$\underline{X}^T \underline{K} \underline{X} = \begin{pmatrix} 3.785 & 0 & 0 \\ 0 & 9.916 & 0 \\ 0 & 0 & 36 \end{pmatrix} \cdot 10^4$$

d)



$\bar{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$a(\omega) = \bar{e}_1^T \left(\frac{\bar{X}_1 \bar{X}_1^T}{(\omega_1^2 - \omega^2)\mu_1} + \frac{\bar{X}_2 \bar{X}_2^T}{(\omega_2^2 - \omega^2)\mu_2} + \frac{\bar{X}_3 \bar{X}_3^T}{(\omega_3^2 - \omega^2)\mu_3} \right) \bar{e}_1$$

$a(\omega) = 0 \Rightarrow \omega = 100, \; 191.5$ (anti-resonans)

e) $x_1 \equiv 0 \Rightarrow$

$T = \frac{1}{2}(3m\dot{x}_2^2 + m\dot{x}_3^2)$

$V = \frac{1}{2}(3kx_2^2 + 2k(x_3 - x_2)^2 + kx_3^2)$

$\underline{M} = \begin{pmatrix} 3m & 0 \\ 0 & m \end{pmatrix}$, $\underline{K} = \begin{pmatrix} 5k & -2k \\ -2k & 3k \end{pmatrix}$

Natural frequencies:

$\det(-\omega^2 \underline{M} + \underline{K}) = 0 \Rightarrow$ (using Mathcad)

$\omega_1^2 = \frac{k}{m} \Rightarrow \omega_1^2 = 10\,000$, $\omega_1 = 100$

$\omega_2^2 = 3.667\frac{k}{m} \Rightarrow \omega_2^2 = 36670$, $\omega_2 = 191.5$

cf. d)

#

MECHANICAL VIBRATIONS

Exercise 3. Damped vibrations

Problem solutions.

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1. $\underline{C} = \alpha \underline{M} + \beta \underline{K} \Rightarrow$

$\underline{C}\,\underline{M}^{-1}\underline{K} = (\alpha \underline{M} + \beta \underline{K})\underline{M}^{-1}\underline{K} = \alpha \underline{K} + \beta \underline{K}\,\underline{M}^{-1}\underline{K}$

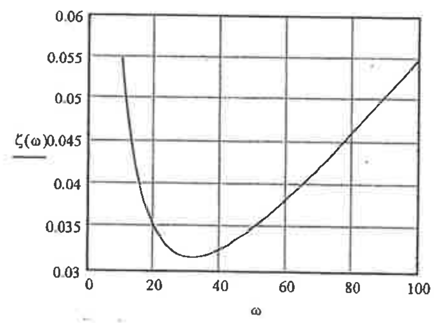
$\underline{K}\,\underline{M}^{-1}\underline{C} = \underline{K}\,\underline{M}^{-1}(\alpha \underline{M} + \beta \underline{K}) = \alpha \underline{K} + \beta \underline{K}\,\underline{M}^{-1}K$

$\Rightarrow \underline{C}\,\underline{M}^{-1}\underline{K} = \underline{K}\,\underline{M}^{-1}\underline{C}$

--- #

2. $\zeta(\omega) = \frac{1}{2}\left(\frac{\alpha}{\omega} + \beta\omega\right) = \frac{1}{2}\left(\frac{1}{\omega} + 0.001\omega\right)$

The plot is shown below:



3. $\underline{M}\ddot{\bar{q}} + \underline{C}\dot{\bar{q}} + \underline{K}\bar{q} = \bar{0}$

$\bar{q}(0) = \bar{q}_0 \,, \quad \dot{\bar{q}}(0) = \dot{\bar{q}}_0$

Using the classical modal matrix $\underline{X}$ and the coordinatetranformation

$\bar{q} = \underline{X}\bar{\eta}$

we get the uncoupled equations

$\mu_i \ddot{\eta}_i + \gamma_i \dot{\eta}_i + \varkappa_i \eta_i = 0$

with solutions $(0 \leq \zeta_i < 1)$

$\eta_i = e^{-\zeta_i \omega_{0i} t}(A_i \cos\omega_{di} t + B_i \sin\omega_{di} t)$

and then

$$\bar{q} = \underline{X}\bar{\eta} = \sum_{i=1}^{n} \bar{x}_i e^{-\zeta_i \omega_{0i} t}(A_i \cos\omega_{di} t + B_i \sin\omega_{di} t) \qquad (3.1)$$

$\bar{q}(0) = \bar{q}_0 \Rightarrow$

$\sum_{i=1}^{n} \bar{x}_i A_i = \bar{q}_0 \Rightarrow A_i = \frac{\bar{x}_i^T \underline{M}\,\bar{q}_0}{\mu_i}$

$\dot{\bar{q}}(0) = \dot{\bar{q}}_0 \Rightarrow$

$$\sum_{i=1}^{n} \bar{x}_i \{(-\zeta_i \omega_{0i}) A_i + \omega_{di} B_i)\} = \dot{\bar{q}}_0 \implies$$

$$\bar{x}_i^T \underline{M} \bar{x}_i (-\zeta_i \omega_{0i} A_i + \omega_{di} B_i) = \bar{x}_i^T \underline{M} \dot{\bar{q}}_0 \implies$$

$$B_i = \frac{\bar{x}_i^T M \dot{\bar{q}}_0}{\omega_{di} \mu_i} + \zeta_i \frac{\omega_{0i}}{\omega_{di}} \frac{\bar{x}_i \underline{M} \bar{q}_0}{\mu_i}$$

Inserting this into (3.1) gives

$$q = [\sum_{i=1}^{n} e^{-\zeta_i \omega_{0i} t} (\frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i} \cos \omega_{di} t +$$

$$\zeta_i \frac{\omega_{0i}}{\omega_{di}} \frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\mu_i} \sin \omega_{di} t)] \bar{q}_0 +$$

$$[\sum_{i=1}^{n} e^{-\zeta_i \omega_{0i} t} \frac{\bar{x}_i \bar{x}_i^T \underline{M}}{\omega_{di} \mu_i}] \dot{\bar{q}}_0$$

#

4. $p(s) = \det(\underline{M} s^2 + \underline{C} s + \underline{K}) =$

$$\det \underline{M} \, s^{2n} + c_{2n-1} s^{2n-1} + \ldots + \det \underline{K} = 0$$

Take the complex conjugate of this equation and use the following rules

$$(z_1 + z_2)^* = z_1^* + z_2^* \, , \quad (z_1 z_2)^* = z_1^* z_2^* \quad z_1, z_2 \in \mathbb{C}$$

$$z^* = z \iff z \in \mathbb{R}$$

we get

$$\det \underline{M} \, (s^*)^{2n} + c_{2n-1} (s^*)^{2n-1} + \ldots + \det \underline{K} = 0$$

i.e. $p(s^*) = 0$

#

5

6

Exercise 3:5

Input:

$m_c := 1$ $m := 1$ $k := 100$

$l := 1$ $g := 9.81$

Calculations:

Mass-, stiffness- and damping-matrices:

$$M := \begin{pmatrix} m_c + m & m\cdot l \\ m\cdot l & m\cdot l^2 \end{pmatrix} \qquad M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$K := \begin{pmatrix} k & 0 \\ 0 & m\cdot g\cdot l \end{pmatrix} \qquad K = \begin{pmatrix} 100 & 0 \\ 0 & 9.81 \end{pmatrix}$$

$$C := 0.7\cdot M \qquad C = \begin{pmatrix} 1.4 & 0.7 \\ 0.7 & 0.7 \end{pmatrix}$$ Rayleigh damping

Undamped eigenfrequencies:

$\omega 2 := \text{genvals}(K, M)$

$$\omega 2 = \begin{pmatrix} 110.763 \\ 8.857 \end{pmatrix} \qquad \sqrt{\omega 2} = \begin{pmatrix} 10.524 \\ 2.976 \end{pmatrix}$$

$$\omega_{0_1} := \left(\sqrt{\omega 2}\right)_2 \qquad \omega_{0_2} := \left(\sqrt{\omega 2}\right)_1 \qquad \omega_0 = \begin{pmatrix} 2.976 \\ 10.524 \end{pmatrix}$$

a) Damped eigenfrequencies:

Impedance matrix:

$Z(s) := M\cdot s^2 + C\cdot s + K$

Characteristicpolynomial:

$p(s) := |Z(s)|$

$p(s) \rightarrow s^4 + 1.4\cdot s^3 + 120.11\cdot s^2 + 83.734\cdot s + 981.00$

Coefficientvector:

$I_4 := |M| \qquad I_4 = 1$

$I_3 := 1.4$

$I_2 := 120.11$

$I_1 := 83.734$

$I_0 := |K| \qquad I_0 = 981$

$$I := \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} \qquad I = \begin{pmatrix} 981 \\ 83.734 \\ 120.11 \\ 1.4 \\ 1 \end{pmatrix}$$

Roots to the characteristic equation:

$$\lambda := \text{polyroots}(I) \qquad \lambda = \begin{pmatrix} -0.350000001006178 + 2.95537245420042i \\ -0.350000000282704 - 10.5185918095985i \\ -0.349999999717299 + 10.5185918095985i \\ -0.34999999899382 - 2.95537245420043i \end{pmatrix}$$

c) Mode shapes

Classical mode shapes:

$U := \text{genvecs}(K, M)$

$$U = \begin{pmatrix} 0.674 & -0.107 \\ -0.739 & -0.994 \end{pmatrix}$$

$$x_1 := \frac{1}{U_{1,2}}\cdot U^{\langle 2\rangle} \qquad x_1 = \begin{pmatrix} 1 \\ 9.291 \end{pmatrix}$$ corresponding to ω'=2.976

$$x_2 := \frac{1}{U_{1,1}}\cdot U^{\langle 1\rangle} \qquad x_2 = \begin{pmatrix} 1 \\ -1.097 \end{pmatrix}$$ corresponding to ω =10.524

Modal matrix: $X^{\langle 1\rangle} := x_1 \qquad X^{\langle 2\rangle} := x_2 \qquad X = \begin{pmatrix} 1 & 1 \\ 9.291 & -1.097 \end{pmatrix}$

Damped mode shapes:

First eigenvalue:

$\lambda_1 = -0.35 + 2.955i$

$\omega_{d1} := \mathrm{Im}(\lambda_1)$ $\omega_{d1} = 2.955$ damped natural frequency

$\omega_{0_1} = 2.976$

$\zeta_1 := \dfrac{|\mathrm{Re}(\lambda_1)|}{\omega_{0_1}}$ $\zeta_1 = 0.118$ relative damping

$S := Z(\lambda_1)$ $S = \begin{pmatrix} 82.287 - 1.189i \times 10^{-8} & -8.857 - 5.947i \times 10^{-9} \\ -8.857 - 5.947i \times 10^{-9} & 0.953 - 5.947i \times 10^{-9} \end{pmatrix}$

The adjoint matrix Q:

$Q_{1,1} := (-1)^{1+1} \cdot S_{2,2}$ $Q_{2,1} := (-1)^{2+1} \cdot S_{1,2}$

$Q_{1,2} := (-1)^{1+2} \cdot S_{2,1}$ $Q_{2,2} := (-1)^{2+2} \cdot S_{1,1}$

$Q = \begin{pmatrix} 0.953 - 5.947i \times 10^{-9} & 8.857 + 5.947i \times 10^{-9} \\ 8.857 + 5.947i \times 10^{-9} & 82.287 - 1.189i \times 10^{-8} \end{pmatrix}$

$S \cdot Q = \begin{pmatrix} -3.64 \times 10^{-6} - 6.061i \times 10^{-7} & 0 \\ 0 & -3.64 \times 10^{-6} - 6.061i \times 10^{-7} \end{pmatrix}$

$\dfrac{Q^{\langle 1 \rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ 9.291 + 6.42i \times 10^{-8} \end{pmatrix}$ $\dfrac{Q^{\langle 2 \rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ 9.291 - 7.582i \times 10^{-9} \end{pmatrix}$

Comparison with the classical normal mode:

$x_1 = \begin{pmatrix} 1 \\ 9.291 \end{pmatrix}$

Second eigenvalue:

$\lambda_2 = -0.35 - 10.519i$

$\omega_{d2} := |\mathrm{Im}(\lambda_2)|$ $\omega_{d2} = 10.519$ damped natural frequency

$\omega_{0_2} = 10.524$ undamped natural frequency

$\zeta_2 := \dfrac{|\mathrm{Re}(\lambda_2)|}{\omega_{0_2}}$ $\zeta_2 = 0.033$ relative damping

$S := Z(\lambda_2)$ $S = \begin{pmatrix} -121.527 & -110.763 \\ -110.763 & -100.953 \end{pmatrix}$

The adjoint matrix:

$Q_{1,1} := (-1)^{1+1} \cdot S_{2,2}$ $Q_{2,1} := (-1)^{2+1} \cdot S_{1,2}$

$Q_{1,2} := (-1)^{1+2} \cdot S_{2,1}$ $Q_{2,2} := (-1)^{2+2} \cdot S_{1,1}$

$Q = \begin{pmatrix} -100.953 & 110.763 \\ 110.763 & -121.527 \end{pmatrix}$

$S \cdot Q = \begin{pmatrix} -3.64 \times 10^{-6} - 6.061i \times 10^{-7} & 0 \\ 0 & -3.64 \times 10^{-6} - 6.061i \times 10^{-7} \end{pmatrix}$

$\dfrac{Q^{\langle 1 \rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ -1.097 \end{pmatrix}$ $\dfrac{Q^{\langle 2 \rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ -1.097 \end{pmatrix}$

Comparison with the classical normal modes:

$x_2 = \begin{pmatrix} 1 \\ -1.097 \end{pmatrix}$

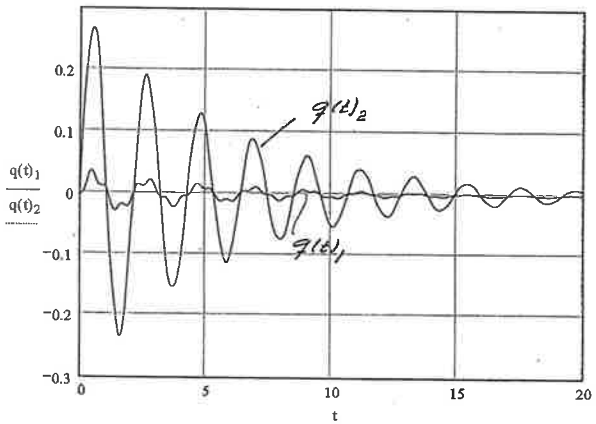
d) Free vibrations:

$q_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad v_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ Initial data

The motion for the damped system (C=0.7M):

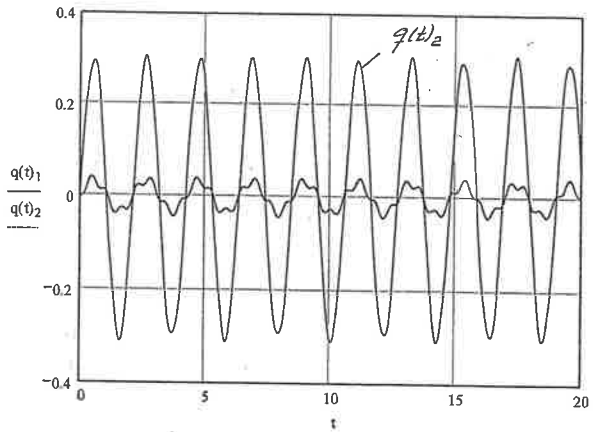
$$q(t) := \left[e^{-\zeta_1 \cdot \omega_{0_1} \cdot t} \cdot \left(\frac{X^{\langle 1\rangle} \cdot \left(X^{\langle 1\rangle}\right)^T \cdot M}{\left(X^{\langle 1\rangle}\right)^T \cdot M \cdot X^{\langle 1\rangle}} \cdot \cos(\omega_{d1} \cdot t) + \zeta_1 \cdot \frac{\omega_{0_1}}{\omega_{d1}} \cdot \frac{X^{\langle 1\rangle} \cdot \left(X^{\langle 1\rangle}\right)^T \cdot M}{\left(X^{\langle 1\rangle}\right)^T \cdot M \cdot X^{\langle 1\rangle}} \cdot \sin(\omega_{d1} \cdot t) \right) \cdots + e^{-\zeta_2 \cdot \omega_{0_2} \cdot t} \cdot \left(\frac{X^{\langle 2\rangle} \cdot \left(X^{\langle 2\rangle}\right)^T \cdot M}{\left(X^{\langle 2\rangle}\right)^T \cdot M \cdot X^{\langle 2\rangle}} \cdot \cos(\omega_{d2} \cdot t) + \zeta_2 \cdot \frac{\omega_{0_2}}{\omega_{d2}} \cdot \frac{X^{\langle 2\rangle} \cdot \left(X^{\langle 2\rangle}\right)^T \cdot M}{\left(X^{\langle 2\rangle}\right)^T \cdot M \cdot X^{\langle 2\rangle}} \cdot \sin(\omega_{d2} \cdot t) \right) \right] \cdot q_0 \cdots$$
$$+ \left[e^{-\zeta_1 \cdot \omega_{0_1} \cdot t} \cdot \frac{X^{\langle 1\rangle} \cdot \left(X^{\langle 1\rangle}\right)^T \cdot M}{\left(\left(X^{\langle 1\rangle}\right)^T \cdot M \cdot X^{\langle 1\rangle}\right) \cdot \omega_{0_1}} \cdot \sin(\omega_{d1} \cdot t) \cdots + e^{-\zeta_2 \cdot \omega_{0_2} \cdot t} \cdot \frac{X^{\langle 2\rangle} \cdot \left(X^{\langle 2\rangle}\right)^T \cdot M}{\left(\left(X^{\langle 2\rangle}\right)^T \cdot M \cdot X^{\langle 2\rangle}\right) \cdot \omega_{0_2}} \cdot \sin(\omega_{d2} \cdot t) \right] \cdot v_0$$

$t := 0, 0.001 .. 20$



The motion for the undamped system (C = 0):

$$q(t) := \left(\frac{X^{\langle 1\rangle} \cdot \left(X^{\langle 1\rangle}\right)^T \cdot M}{\left(X^{\langle 1\rangle}\right)^T \cdot M \cdot X^{\langle 1\rangle}} \cdot \cos(\omega_{0_1} \cdot t) \cdots + \frac{X^{\langle 2\rangle} \cdot \left(X^{\langle 2\rangle}\right)^T \cdot M}{\left(X^{\langle 2\rangle}\right)^T \cdot M \cdot X^{\langle 2\rangle}} \cdot \cos(\omega_{0_2} \cdot t) \right) \cdot q_0 \cdots$$
$$+ \left[\frac{X^{\langle 1\rangle} \cdot \left(X^{\langle 1\rangle}\right)^T \cdot M}{\left(\left(X^{\langle 1\rangle}\right)^T \cdot M \cdot X^{\langle 1\rangle}\right) \cdot \omega_{0_1}} \cdot \sin(\omega_{0_1} \cdot t) \cdots + \frac{X^{\langle 2\rangle} \cdot \left(X^{\langle 2\rangle}\right)^T \cdot M}{\left(\left(X^{\langle 2\rangle}\right)^T \cdot M \cdot X^{\langle 2\rangle}\right) \cdot \omega_{0_2}} \cdot \sin(\omega_{0_2} \cdot t) \right] \cdot v_0$$



Exercise 3:6

Input:

$m_1 := 0.5$

$l := 2.0$

$k := 10$

$c := 0.5$

$g := 9.81$

Calculations:

a) Mass- and stiffness- and damping-matrices:

$$M := \begin{pmatrix} m_1 \cdot l^2 & 0 \\ 0 & m_1 \cdot l^2 \end{pmatrix} \qquad K := \begin{pmatrix} m_1 \cdot g \cdot l + \frac{1}{4} \cdot k \cdot l^2 & \frac{1}{4} \cdot k \cdot l^2 \\ \frac{1}{4} \cdot k \cdot l^2 & m_1 \cdot g \cdot l + \frac{1}{4} \cdot k \cdot l^2 \end{pmatrix}$$

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad K = \begin{pmatrix} 19.81 & 10 \\ 10 & 19.81 \end{pmatrix} \qquad C := \begin{pmatrix} c \cdot \frac{l^2}{4} & -c \cdot \frac{l^2}{4} \\ -c \cdot \frac{l^2}{4} & c \cdot \frac{l^2}{4} \end{pmatrix}$$

b) Is the system diagonalizable?

$$C \cdot M^{-1} \cdot K = \begin{pmatrix} 2.453 & -2.453 \\ -2.453 & 2.453 \end{pmatrix} \qquad K \cdot M^{-1} \cdot C = \begin{pmatrix} 2.453 & -2.453 \\ -2.453 & 2.453 \end{pmatrix}$$

Yes!

Undamped eigenfrequencies:

$$\Omega 2 := \mathrm{genvals}(K, M) \qquad \Omega 2 = \begin{pmatrix} 14.905 \\ 4.905 \end{pmatrix}$$

$$\omega 2 := \begin{pmatrix} \Omega 2_2 \\ \Omega 2_1 \end{pmatrix} \qquad \omega 2 = \begin{pmatrix} 4.905 \\ 14.905 \end{pmatrix} \qquad \omega_0 := \sqrt{\omega 2} \qquad \omega_0 = \begin{pmatrix} 2.215 \\ 3.861 \end{pmatrix}$$

Classical normal mode shapes:

$$U := \mathrm{genvecs}(K, M)$$

$$U = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

$$x_1 := \frac{1}{U_{1,2}} \cdot U^{\langle 2 \rangle} \qquad x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$x_2 := \frac{1}{U_{1,1}} \cdot U^{\langle 1 \rangle} \qquad x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Modal matrix: $X^{\langle 1 \rangle} := x_1 \qquad X^{\langle 2 \rangle} := x_2 \qquad X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

Damped eigenfrequencies:

Impedance matrix:

$$Z(s) := M \cdot s^2 + C \cdot s + K$$

Characteristic polynomial:

$$p(s) := |Z(s)|$$

$$p(s) \rightarrow 4.000000 \cdot s^4 + 2.0000000000000000000 \cdot s^3 + 79.240000000000000000 \cdot s^2 + 29.81000000 \cdot s + 292.4361000$$

Coefficientvector

$I_4 := |M| \qquad I_4 = 4$

$I_3 := 2.0$

$I_2 := 79.24$

$I_1 := 29.81$

$I_0 := |K| \qquad I_0 = 292.436$

$$I := \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} \qquad I = \begin{pmatrix} 292.436 \\ 29.81 \\ 79.24 \\ 2 \\ 4 \end{pmatrix}$$

Roots to the characteristic equation:

$\lambda :=$ polyroots(I)

$$\lambda = \begin{pmatrix} -0.25 + 2.20056810846654i \\ -0.24999992489505 - 2.20056833197342i \\ -5.17204862802922 \times 10^{-8} - 3.86069924061546i \\ -2.33844638399051 \times 10^{-8} + 3.86069946412234i \end{pmatrix}$$

Damped mode shapes:

First eigenvalue:

$\lambda_1 = -0.25 + 2.201i$

$\omega_{d1} := \mathrm{Im}(\lambda_1) \qquad \omega_{d1} = 2.201$ damped natural frequency

$\omega_{0_1} = 2.215$

$\zeta_1 := \dfrac{|\mathrm{Re}(\lambda_1)|}{\omega_{0_1}} \qquad \zeta_1 = 0.113$ relative damping

$S := Z(\lambda_1) \qquad S = \begin{pmatrix} 10.125 - 1.1i & 10.125 - 1.1i \\ 10.125 - 1.1i & 10.125 - 1.1i \end{pmatrix}$

The adjoint matrix Q:

$Q_{1,1} := (-1)^{1+1} \cdot S_{2,2} \qquad Q_{2,1} := (-1)^{2+1} \cdot S_{1,2}$

$Q_{1,2} := (-1)^{1+2} \cdot S_{2,1} \qquad Q_{2,2} := (-1)^{2+2} \cdot S_{1,1}$

$Q = \begin{pmatrix} 10.125 - 1.1i & -10.125 + 1.1i \\ -10.125 + 1.1i & 10.125 - 1.1i \end{pmatrix}$

$S \cdot Q = \begin{pmatrix} -1.429 \times 10^{-13} + 2.463i \times 10^{-14} & 0 \\ 0 & -1.429 \times 10^{-13} + 2.463i \times 10^{-14} \end{pmatrix}$

$\dfrac{Q^{\langle 1 \rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \dfrac{Q^{\langle 2 \rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Comparison with the classical normal mode:

$x_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Second eigenvalue:

$\lambda_3 = -5.172 \times 10^{-8} - 3.861i$

$\omega_{d2} := |\mathrm{Im}(\lambda_3)| \qquad \omega_{d2} = 3.861$ damped natural frequency

$\omega_{0_2} = 3.861$ undamped natural frequency

$\zeta_2 := \dfrac{|\mathrm{Re}(\lambda_3)|}{\omega_{0_2}} \qquad \zeta_2 = 1.34 \times 10^{-8}$ relative damping

$S := Z(\lambda_3) \qquad S = \begin{pmatrix} -10 - 1.93i & 10 + 1.93i \\ 10 + 1.93i & -10 - 1.93i \end{pmatrix}$

The adjoint matrix:

$Q_{1,1} := (-1)^{1+1} \cdot S_{2,2} \qquad Q_{2,1} := (-1)^{2+1} \cdot S_{1,2}$

$Q_{1,2} := (-1)^{1+2} \cdot S_{2,1} \qquad Q_{2,2} := (-1)^{2+2} \cdot S_{1,1}$

$Q = \begin{pmatrix} -10 - 1.93i & -10 - 1.93i \\ -10 - 1.93i & -10 - 1.93i \end{pmatrix}$

$S \cdot Q = \begin{pmatrix} -5.186 \times 10^{-5} - 2.658i \times 10^{-5} & 0 \\ 0 & -5.186 \times 10^{-5} - 2.658i \times 10^{-5} \end{pmatrix}$

$\dfrac{Q^{\langle 1 \rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ 1 + 2.588i \times 10^{-8} \end{pmatrix} \qquad \dfrac{Q^{\langle 2 \rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ 1 - 2.588i \times 10^{-8} \end{pmatrix}$

Comparison with the classical normal modes:

$x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Exercise 3:7

Input:

$m_1 := 1$

$k := 1000$

$g := 9.81$

Calculations:

Mass- and stiffness- and damping-matrices:

$M := \begin{pmatrix} m_1 & 0 \\ 0 & m_1 \end{pmatrix}$ $\quad M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$K := \begin{pmatrix} 2\cdot k & -k \\ -k & 2\cdot k \end{pmatrix}$ $\quad K = \begin{pmatrix} 2\times 10^3 & -1\times 10^3 \\ -1\times 10^3 & 2\times 10^3 \end{pmatrix}$

$C := 0.006\cdot K$ $\quad C = \begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix}$

b) Undamped eigenfrequencies:

$\Omega 2 := \mathrm{genvals}(K, M)$

$\Omega 2 = \begin{pmatrix} 3\times 10^3 \\ 1\times 10^3 \end{pmatrix}$

$\omega 2 := \begin{pmatrix} \Omega 2_2 \\ \Omega 2_1 \end{pmatrix}$ $\quad \omega 2 = \begin{pmatrix} 1\times 10^3 \\ 3\times 10^3 \end{pmatrix}$ $\quad \omega_0 := \sqrt{\omega 2}$ $\quad \omega_0 = \begin{pmatrix} 31.623 \\ 54.772 \end{pmatrix}$

Classical normal mode shapes:

$U := \mathrm{genvecs}(K, M)$

$U = \begin{pmatrix} 0.707 & -0.707 \\ -0.707 & -0.707 \end{pmatrix}$

$x_1 := \frac{1}{U_{1,2}}\cdot U^{\langle 2\rangle}$ $\quad x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x_2 := \frac{1}{U_{1,1}}\cdot U^{\langle 1\rangle}$ $\quad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Modal matrix: $X^{\langle 1\rangle} := x_1$ $\quad X^{\langle 2\rangle} := x_2$ $\quad X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Damped eigenfrequencies:

Impedance matrix:

$Z(s) := M\cdot s^2 + C\cdot s + K$

Characteristicpolynomial:

$p(s) := |Z(s)|$

$p(s) \rightarrow s^4 + 24.000\cdot s^3 + 4108.000000\cdot s^2 + 36000.000\cdot s + 3000000$

Coefficient vector:

$I_4 := |M|$ $\quad I_4 = 1$

$I_3 := 24.0$

$I_2 := 4108.0$

$I_1 := 36000.0$

$I_0 := |K|$ $\quad I_0 = 3\times 10^6$

$I := \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix}$ $\quad I = \begin{pmatrix} 3\times 10^6 \\ 3.6\times 10^4 \\ 4.108\times 10^3 \\ 24 \\ 1 \end{pmatrix}$

Roots to the characteristic equation:

$\lambda := \mathrm{polyroots}(I)$ $\quad \lambda = \begin{pmatrix} -9.00000029500415 - 54.02777308416961i \\ -9.00000029189855 + 54.02777003283711i \\ -3 + 31.4801524773944i \\ -2.99999941309731 - 31.48014942606191i \end{pmatrix}$

Damped mode shapes:

First eigenvalue:

$\lambda_3 = -3 + 31.48i$

$\omega_{d1} := \mathrm{Im}(\lambda_3)$ $\quad \omega_{d1} = 31.48$ $\quad$ damped natural frequency

$\omega_{0_1} = 31.623$

$\zeta_1 := \frac{|Re(\lambda_3)|}{\omega_{0_1}}$ $\zeta_1 = 0.095$ relative damping

$S := Z(\lambda_3)$ $S = \begin{pmatrix} 982 + 188.881i & -982 - 188.881i \\ -982 - 188.881i & 982 + 188.881i \end{pmatrix}$

The adjoint matrix Q:

$Q_{1,1} := (-1)^{1+1} \cdot S_{2,2}$ $Q_{2,1} := (-1)^{2+1} \cdot S_{1,2}$

$Q_{1,2} := (-1)^{1+2} \cdot S_{2,1}$ $Q_{2,2} := (-1)^{2+2} \cdot S_{1,1}$

$Q = \begin{pmatrix} 982 + 188.881i & 982 + 188.881i \\ 982 + 188.881i & 982 + 188.881i \end{pmatrix}$

$S \cdot Q = \begin{pmatrix} 4.358 \times 10^{-10} + 1.417i \times 10^{-10} & 0 \\ 0 & 4.358 \times 10^{-10} + 1.417i \times 10^{-10} \end{pmatrix}$

$\frac{Q^{\langle 1 \rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\frac{Q^{\langle 2 \rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Comparison with the classical normal mode:

$x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Second eigenvalue:

$\lambda_2 = -9 + 54.028i$

$\omega_{d2} := Im(\lambda_2)$ $\omega_{d2} = 54.028$ damped natural frequency

$\omega_{0_2} = 54.772$ undamped natural frequency

$\zeta_2 := \frac{|Re(\lambda_2)|}{\omega_{0_2}}$ $\zeta_2 = 0.164$ relative damping

$S := Z(\lambda_2)$ $S = \begin{pmatrix} -946 - 324.167i & -946 - 324.167i \\ -946 - 324.167i & -946 - 324.167i \end{pmatrix}$

$\zeta := \sqrt{\frac{1}{1 + \left(\frac{Im(\lambda_2)}{Re(\lambda_2)}\right)^2}}$ $\zeta = 0.164$

The adjoint matrix:

$Q_{1,1} := (-1)^{1+1} \cdot S_{2,2}$ $Q_{2,1} := (-1)^{2+1} \cdot S_{1,2}$

$Q_{1,2} := (-1)^{1+2} \cdot S_{2,1}$ $Q_{2,2} := (-1)^{2+2} \cdot S_{1,1}$

$Q = \begin{pmatrix} -946 - 324.167i & 946 + 324.167i \\ 946 + 324.167i & -946 - 324.167i \end{pmatrix}$

$S \cdot Q = \begin{pmatrix} -0.144 + 0.017i & 0 \\ 0 & -0.144 + 0.017i \end{pmatrix}$

$\frac{Q^{\langle 1 \rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ -1 + 5.1i \times 10^{-8} \end{pmatrix}$ $\frac{Q^{\langle 2 \rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ -1 - 5.1i \times 10^{-8} \end{pmatrix}$

Comparison with the classical normal modes:

$x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) Forced vibrations:

$$q_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad v_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ Initial data

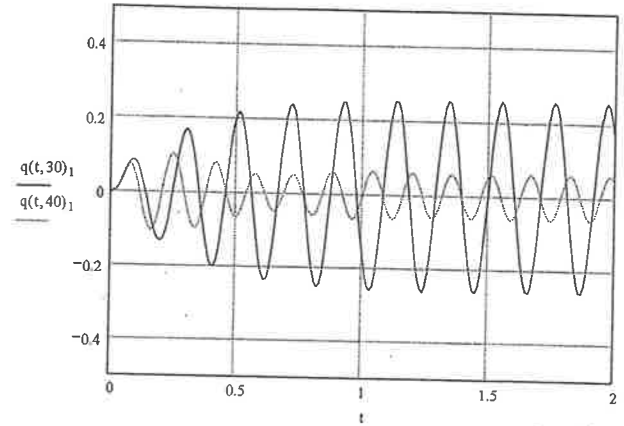
$$\sigma := \begin{pmatrix} 100 \\ 0 \end{pmatrix}$$ Load amplitude

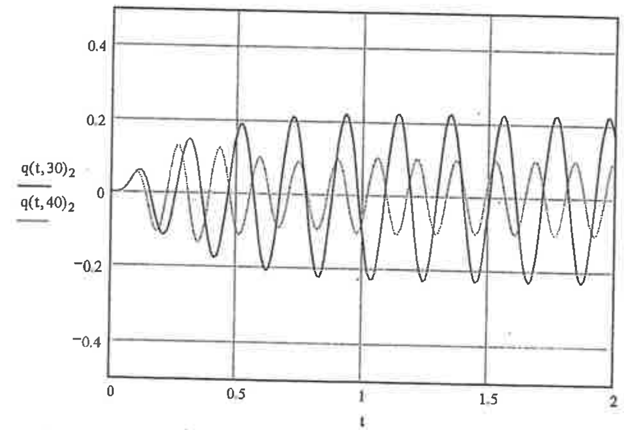
$$\omega := 30$$ Load frequency

The motion for the damped system:

$$\begin{aligned} q(t,\omega) := & \left[e^{-\zeta_1\cdot\omega_{0_1}\cdot t}\cdot\left(\frac{X^{\langle 1\rangle}\cdot\left(X^{\langle 1\rangle}\right)^T\cdot M}{\left(X^{\langle 1\rangle}\right)^T\cdot M\cdot X^{\langle 1\rangle}}\cdot\cos(\omega_{d1}\cdot t) + \zeta_1\cdot\frac{\omega_{0_1}}{\omega_{d1}}\cdot\frac{X^{\langle 1\rangle}\cdot\left(X^{\langle 1\rangle}\right)^T\cdot M}{X^{\langle 1\rangle T}\cdot M\cdot X^{\langle 1\rangle}}\cdot\sin(\omega_{d1}\cdot t)\right) \dots \right. \\ & \left. + e^{-\zeta_2\cdot\omega_{0_2}\cdot t}\cdot\left(\frac{X^{\langle 2\rangle}\cdot\left(X^{\langle 2\rangle}\right)^T\cdot M}{\left(X^{\langle 2\rangle}\right)^T\cdot M\cdot X^{\langle 2\rangle}}\cdot\cos(\omega_{d2}\cdot t) + \zeta_2\cdot\frac{\omega_{0_2}}{\omega_{d2}}\cdot\frac{X^{\langle 2\rangle}\cdot\left(X^{\langle 2\rangle}\right)^T\cdot M}{\left(X^{\langle 2\rangle}\right)^T\cdot M\cdot X^{\langle 2\rangle}}\cdot\sin(\omega_{d2}\cdot t)\right) \right]\cdot q_0 \dots \\ & + \left[e^{-\zeta_1\cdot\omega_{0_1}\cdot t}\cdot\frac{X^{\langle 1\rangle}\cdot\left(X^{\langle 1\rangle}\right)^T\cdot M}{\left(\left(X^{\langle 1\rangle}\right)^T\cdot M\cdot X^{\langle 1\rangle}\right)\cdot\omega_{d1}}\cdot\sin(\omega_{d1}\cdot t) \dots \right. \\ & \left. + e^{-\zeta_2\cdot\omega_{0_2}\cdot t}\cdot\frac{X^{\langle 2\rangle}\cdot\left(X^{\langle 2\rangle}\right)^T\cdot M}{\left(\left(X^{\langle 2\rangle}\right)^T\cdot M\cdot X^{\langle 2\rangle}\right)\cdot\omega_{d2}}\cdot\sin(\omega_{d2}\cdot t) \right]\cdot v_0 \dots \\ & + \int_0^t e^{-\zeta_1\cdot\omega_{0_1}\cdot(t-s)}\cdot\sin\left[\omega_{d1}\cdot(t-s)\right]\cdot\sin(\omega\cdot s)\,ds\cdot\frac{X^{\langle 1\rangle}\cdot\left(X^{\langle 1\rangle}\right)^T\cdot\sigma}{\left(\left(X^{\langle 1\rangle}\right)^T\cdot M\cdot X^{\langle 1\rangle}\right)\cdot\omega_{d1}} \dots \\ & + \int_0^t e^{-\zeta_2\cdot\omega_{0_2}\cdot(t-s)}\cdot\sin\left[\omega_{d2}\cdot(t-s)\right]\cdot\sin(\omega\cdot s)\,ds\cdot\frac{X^{\langle 2\rangle}\cdot\left(X^{\langle 2\rangle}\right)^T\cdot\sigma}{\left(\left(X^{\langle 2\rangle}\right)^T\cdot M\cdot X^{\langle 2\rangle}\right)\cdot\omega_{d2}} \end{aligned}$$

$$t := 0, 0.01 .. 3$$





d) Forced vibrations:

$q_0 := \begin{pmatrix} -0.1 \\ 0.1 \end{pmatrix}$ $\quad v_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad$ Initial data

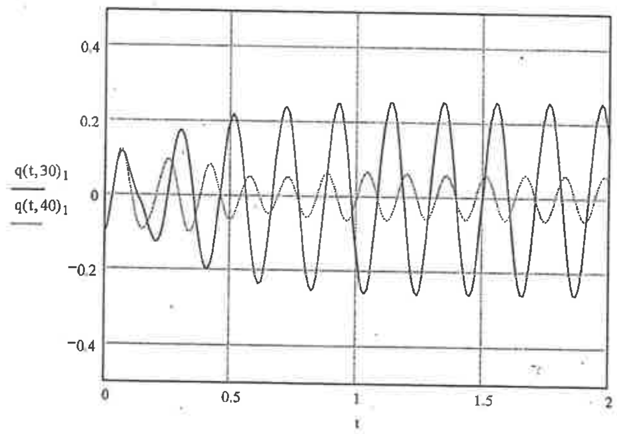
$\sigma := \begin{pmatrix} 100 \\ 0 \end{pmatrix}$ $\quad$ Load amplitude

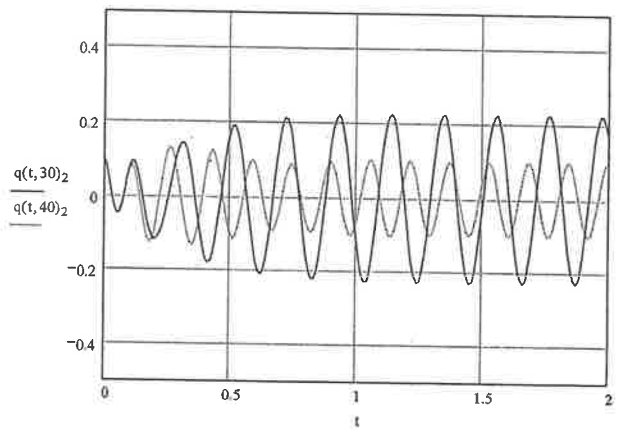
$\omega := 30$ $\quad$ Load frequency

The motion for the damped system:

$$
\begin{aligned}
q(t,\omega) := & \left[e^{-\zeta_1 \cdot \omega_{0_1} \cdot t} \cdot \left(\frac{X^{\langle 1 \rangle} \cdot \left(X^{\langle 1 \rangle}\right)^T \cdot M}{\left(X^{\langle 1 \rangle}\right)^T \cdot M \cdot X^{\langle 1 \rangle}} \cdot \cos(\omega_{d1} \cdot t) + \zeta_1 \cdot \frac{\omega_{0_1}}{\omega_{d1}} \cdot \frac{X^{\langle 1 \rangle} \cdot \left(X^{\langle 1 \rangle}\right)^T \cdot M}{X^{\langle 1 \rangle T} \cdot M \cdot X^{\langle 1 \rangle}} \cdot \sin(\omega_{d1} \cdot t) \right) \ldots \right. \\
& \left. + e^{-\zeta_2 \cdot \omega_{0_2} \cdot t} \cdot \left(\frac{X^{\langle 2 \rangle} \cdot \left(X^{\langle 2 \rangle}\right)^T \cdot M}{\left(X^{\langle 2 \rangle}\right)^T \cdot M \cdot X^{\langle 2 \rangle}} \cdot \cos(\omega_{d2} \cdot t) + \zeta_2 \cdot \frac{\omega_{0_2}}{\omega_{d2}} \cdot \frac{X^{\langle 2 \rangle} \cdot \left(X^{\langle 2 \rangle}\right)^T \cdot M}{\left(X^{\langle 2 \rangle}\right)^T \cdot M \cdot X^{\langle 2 \rangle}} \cdot \sin(\omega_{d2} \cdot t) \right) \right] \cdot q_0 \cdot \\
& + \left[e^{-\zeta_1 \cdot \omega_{0_1} \cdot t} \cdot \frac{X^{\langle 1 \rangle} \cdot \left(X^{\langle 1 \rangle}\right)^T \cdot M}{\left(\left(X^{\langle 1 \rangle}\right)^T \cdot M \cdot X^{\langle 1 \rangle}\right) \cdot \omega_{d1}} \cdot \sin(\omega_{d1} \cdot t) \ldots \right. \\
& \left. + e^{-\zeta_2 \cdot \omega_{0_2} \cdot t} \cdot \frac{X^{\langle 2 \rangle} \cdot \left(X^{\langle 2 \rangle}\right)^T \cdot M}{\left(\left(X^{\langle 2 \rangle}\right)^T \cdot M \cdot X^{\langle 2 \rangle}\right) \cdot \omega_{d2}} \cdot \sin(\omega_{d2} \cdot t) \right] \cdot v_0 \ldots \\
& + \int_0^t e^{-\zeta_1 \cdot \omega_{0_1} \cdot (t-s)} \cdot \sin[\omega_{d1} \cdot (t-s)] \cdot \sin(\omega \cdot s)\, ds \cdot \frac{X^{\langle 1 \rangle} \cdot \left(X^{\langle 1 \rangle}\right)^T \cdot \sigma}{\left(\left(X^{\langle 1 \rangle}\right)^T \cdot M \cdot X^{\langle 1 \rangle}\right) \cdot \omega_{d1}} \ldots \\
& + \int_0^t e^{-\zeta_2 \cdot \omega_{0_2} \cdot (t-s)} \cdot \sin[\omega_{d2} \cdot (t-s)] \cdot \sin(\omega \cdot s)\, ds \cdot \frac{X^{\langle 2 \rangle} \cdot \left(X^{\langle 2 \rangle}\right)^T \cdot \sigma}{\left(\left(X^{\langle 2 \rangle}\right)^T \cdot M \cdot X^{\langle 2 \rangle}\right) \cdot \omega_{d2}}
\end{aligned}
$$

$t := 0, 0.01 .. 3$





Exercise 3:8

Input:

$m := 1 \qquad k := 10000$

Calculations

a) Mass- and stiffness- and damping-matrices:

$$M := \begin{pmatrix} 2m & 0 & 0 \\ 0 & 3\cdot m & 0 \\ 0 & 0 & m \end{pmatrix} \qquad M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$K := \begin{pmatrix} 5\cdot k & -3\cdot k & 0 \\ -3\cdot k & 5\cdot k & -2\cdot k \\ 0 & -2\cdot k & 3\cdot k \end{pmatrix} \qquad K = \begin{pmatrix} 5\times 10^4 & -3\times 10^4 & 0 \\ -3\times 10^4 & 5\times 10^4 & -2\times 10^4 \\ 0 & -2\times 10^4 & 3\times 10^4 \end{pmatrix}$$

$$C := 0.001\cdot K \qquad C = \begin{pmatrix} 50 & -30 & 0 \\ -30 & 50 & -20 \\ 0 & -20 & 30 \end{pmatrix}$$

b) Undamped eigenfrequencies:

$$\Omega2 := \mathrm{genvals}(K, M) \qquad \Omega2 = \begin{pmatrix} 2.741\times 10^4 \\ 4.256\times 10^3 \\ 4\times 10^4 \end{pmatrix}$$

$$\omega2 := \begin{pmatrix} \Omega2_2 \\ \Omega2_1 \\ \Omega2_3 \end{pmatrix} \qquad \omega2 = \begin{pmatrix} 4.256\times 10^3 \\ 2.741\times 10^4 \\ 4\times 10^4 \end{pmatrix} \qquad \omega_0 := \sqrt{\omega2} \qquad \omega_0 = \begin{pmatrix} 65.24 \\ 165.561 \\ 200 \end{pmatrix}$$

Classical normal mode shapes:

$$U := \mathrm{genvecs}(K, M) \qquad U = \begin{pmatrix} -0.624 & -0.496 & 0.408 \\ 0.1 & -0.686 & -0.408 \\ 0.775 & -0.533 & 0.816 \end{pmatrix}$$

$$x_1 := \frac{1}{U_{1,2}}\cdot U^{\langle 2\rangle} \qquad x_1 = \begin{pmatrix} 1 \\ 1.383 \\ 1.074 \end{pmatrix}$$

$$x_2 := \frac{1}{U_{1,1}}\cdot U^{\langle 1\rangle} \qquad x_2 = \begin{pmatrix} 1 \\ -0.161 \\ -1.241 \end{pmatrix}$$

$$x_3 := \frac{1}{U_{1,3}}\cdot U^{\langle 3\rangle} \qquad x_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

b) Damped eigenfrequencies:

Impedance matrix:

$$Z(s) := M\cdot s^2 + C\cdot s + K$$

Characteristicpolynomial:

$$p(s) := |Z(s)|$$

$$p(s) \rightarrow 6\cdot s^6 + 430.000\cdot s^5 + 438300.000000\cdot s^4 + 16628000.000000000\cdot s^3 + 8384000000.000000\cdot s^2 + 84000000000.000\cdot s + 28000000000000$$

Coefficient vector:

$I_6 := |M| \qquad I_6 = 6$

$I_5 := 430$

$I_4 := 438300$

$I_3 := 16628000$

$I_2 := 8384000000.000000$

$I_1 := 84000000000.000$

$I_0 := |K| \qquad I_0 = 2.8\times 10^{13}$

$$I := \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix} \qquad I = \begin{pmatrix} 2.8\times 10^{13} \\ 8.4\times 10^{10} \\ 8.384\times 10^9 \\ 1.663\times 10^7 \\ 4.383\times 10^5 \\ 430 \\ 6 \end{pmatrix}$$

Roots to the characteristic equation:

$\lambda := \mathrm{polyroots}(I)$

$$\lambda = \begin{pmatrix} -20.0000008676844 + 198.997494429147i \\ -19.9999965190587 - 198.997493404432i \\ -13.7051889461608 - 164.99253166617i \\ -13.7051836703884 + 164.992530456527i \\ -2.12814833772925 + 65.205579976802i \\ -2.12814832564526 - 65.205579791875i \end{pmatrix}$$

b) Damped mode shapes:

First eigenvalue:

$\Lambda := \lambda_5$ $\quad\Lambda = -2.128 + 65.206i$

$\omega_{d_1} := \mathrm{Im}(\Lambda)$ $\quad\omega_{d_1} = 65.206$ $\quad$ damped natural frequency

$\omega_{0_1} = 65.24$

$\zeta_1 := \dfrac{|\mathrm{Re}(\Lambda)|}{\omega_{0_1}}$ $\quad\zeta_1 = 0.033$ $\quad$ relative damping

$S := Z(\Lambda)$

$$S = \begin{pmatrix} 4.14\times10^{4} + 2.705i\times10^{3} & -2.994\times10^{4} - 1.956i\times10^{3} & 0 \\ -2.994\times10^{4} - 1.956i\times10^{3} & 3.715\times10^{4} + 2.428i\times10^{3} & -1.996\times10^{4} - 1.304i\times10^{3} \\ 0 & -1.996\times10^{4} - 1.304i\times10^{3} & 2.569\times10^{4} + 1.679i\times10^{3} \end{pmatrix}$$

The adjoint matrix Q:

$Q_{1,1} := (-1)^{1+1}\cdot(S_{2,2}\cdot S_{3,3} - S_{2,3}\cdot S_{3,2})$ $\quad Q_{2,1} := (-1)^{2+1}\cdot(S_{1,2}\cdot S_{3,3} - S_{1,3}\cdot S_{3,2})$

$Q_{1,2} := (-1)^{1+2}\cdot(S_{2,1}\cdot S_{3,3} - S_{2,3}\cdot S_{3,1})$ $\quad Q_{2,2} := (-1)^{2+2}\cdot(S_{1,1}\cdot S_{3,3} - S_{1,3}\cdot S_{3,1})$

$Q_{1,3} := (-1)^{1+3}\cdot(S_{2,1}\cdot S_{3,2} - S_{2,2}\cdot S_{3,1})$ $\quad Q_{2,3} := (-1)^{2+3}\cdot(S_{1,1}\cdot S_{3,2} - S_{1,2}\cdot S_{3,1})$

$Q_{3,1} := (-1)^{3+1}\cdot(S_{1,2}\cdot S_{2,3} - S_{1,3}\cdot S_{2,2})$

$Q_{3,2} := (-1)^{3+2}\cdot(S_{1,1}\cdot S_{2,3} - S_{2,1}\cdot S_{1,3})$

$Q_{3,3} := (-1)^{3+3}\cdot(S_{1,1}\cdot S_{2,2} - S_{1,2}\cdot S_{2,1})$

$$Q = \begin{pmatrix} 5.537\times10^{8} + 7.268i\times10^{7} & 7.657\times10^{8} + 1.005i\times10^{8} & 5.949\times10^{8} + 7.808i\times10^{7} \\ 7.657\times10^{8} + 1.005i\times10^{8} & 1.059\times10^{9} + 1.39i\times10^{8} & 8.227\times10^{8} + 1.08i\times10^{8} \\ 5.949\times10^{8} + 7.808i\times10^{7} & 8.227\times10^{8} + 1.08i\times10^{8} & 6.391\times10^{8} + 8.389i\times10^{7} \end{pmatrix}$$

$$\frac{Q^{\langle 1\rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ 1.383 \\ 1.074 \end{pmatrix} \qquad \frac{Q^{\langle 2\rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ 1.383 \\ 1.074 \end{pmatrix} \qquad \frac{Q^{\langle 3\rangle}}{Q_{1,3}} = \begin{pmatrix} 1 \\ 1.383 \\ 1.074 \end{pmatrix}$$

Comparison with the classical normal mode:

$$x_1 = \begin{pmatrix} 1 \\ 1.383 \\ 1.074 \end{pmatrix}$$

Second eigenvalue:

$\Lambda := \lambda_3$ $\quad\Lambda = -13.705 - 164.993i$

$\omega_{d_2} := \mathrm{Im}(\Lambda)$ $\quad\omega_{d_2} = -164.993$ $\quad$ damped natural frequency

$\omega_{0_2} = 165.561$

$\zeta_2 := \dfrac{|\mathrm{Re}(\Lambda)|}{\omega_{0_2}}$ $\quad\zeta_2 = 0.083$ $\quad$ relative damping

$S := Z(\Lambda)$

$$S = \begin{pmatrix} -4.755\times10^{3} + 795.389i & -2.959\times10^{4} + 4.95i\times10^{3} & 0 \\ -2.959\times10^{4} + 4.95i\times10^{3} & -3.179\times10^{4} + 5.318i\times10^{3} & -1.973\times10^{4} + 3.3i\times10^{3} \\ 0 & -1.973\times10^{4} + 3.3i\times10^{3} & 2.554\times10^{3} - 427.268i \end{pmatrix}$$

The adjoint matrix Q:

$Q_{1,1} := (-1)^{1+1}\cdot(S_{2,2}\cdot S_{3,3} - S_{2,3}\cdot S_{3,2})$ $\quad Q_{2,1} := (-1)^{2+1}\cdot(S_{1,2}\cdot S_{3,3} - S_{1,3}\cdot S_{3,2})$

$Q_{1,2} := (-1)^{1+2}\cdot(S_{2,1}\cdot S_{3,3} - S_{2,3}\cdot S_{3,1})$ $\quad Q_{2,2} := (-1)^{2+2}\cdot(S_{1,1}\cdot S_{3,3} - S_{1,3}\cdot S_{3,1})$

$Q_{1,3} := (-1)^{1+3}\cdot(S_{2,1}\cdot S_{3,2} - S_{2,2}\cdot S_{3,1})$ $\quad Q_{2,3} := (-1)^{2+3}\cdot(S_{1,1}\cdot S_{3,2} - S_{1,2}\cdot S_{3,1})$

$$Q_{3,1} := (-1)^{3+1}\cdot(S_{1,2}\cdot S_{2,3} - S_{1,3}\cdot S_{2,2})$$

$$Q_{3,2} := (-1)^{3+2}\cdot(S_{1,1}\cdot S_{2,3} - S_{2,1}\cdot S_{1,3})$$

$$Q_{3,3} := (-1)^{3+3}\cdot(S_{1,1}\cdot S_{2,2} - S_{1,2}\cdot S_{2,1})$$

$$Q = \begin{pmatrix} -4.571\times10^{8} + 1.574i\times10^{8} & 7.346\times10^{7} - 2.528i\times10^{7} & 5.673\times10^{8} - 1.953i\times10^{8} \\ 7.346\times10^{7} - 2.528i\times10^{7} & -1.18\times10^{7} + 4.063i\times10^{6} & -9.117\times10^{7} + 3.138i\times10^{7} \\ 5.673\times10^{8} - 1.953i\times10^{8} & -9.117\times10^{7} + 3.138i\times10^{7} & -7.041\times10^{8} + 2.423i\times10^{8} \end{pmatrix}$$

$$\frac{Q^{\langle 1\rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ -0.161 - 9.059i\times10^{-8} \\ -1.241 + 1.055i\times10^{-7} \end{pmatrix} \qquad \frac{Q^{\langle 2\rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ -0.161 + 1.12i\times10^{-7} \\ -1.241 + 1.67i\times10^{-6} \end{pmatrix}$$

$$\frac{Q^{\langle 3\rangle}}{Q_{1,3}} = \begin{pmatrix} 1 \\ -0.161 + 1.12i\times10^{-7} \\ -1.241 - 2.21i\times10^{-7} \end{pmatrix}$$

Comparison with the classical normal modes:

$$x_2 = \begin{pmatrix} 1 \\ -0.161 \\ -1.241 \end{pmatrix}$$

Third eigenvalue:

$\Lambda := \lambda_1$ $\qquad \Lambda = -20 + 198.997i$

$\omega_{d_3} := \mathrm{Im}(\Lambda)$ $\qquad \omega_{d_3} = 198.997$ $\qquad$ damped natural frequency

$\omega_{0_3} = 200$

$\zeta_3 := \dfrac{|\mathrm{Re}(\Lambda)|}{\omega_{0_3}}$ $\qquad \zeta_3 = 0.1$ $\qquad$ relative damping

$S := Z(\Lambda)$

$$S = \begin{pmatrix} -2.94\times10^{4} - 5.97i\times10^{3} & -2.94\times10^{4} - 5.97i\times10^{3} & 0 \\ -2.94\times10^{4} - 5.97i\times10^{3} & -6.86\times10^{4} - 1.393i\times10^{4} & -1.96\times10^{4} - 3.98i\times10^{3} \\ 0 & -1.96\times10^{4} - 3.98i\times10^{3} & -9.8\times10^{3} - 1.99i\times10^{3} \end{pmatrix}$$

The adjoint matrix Q:

$$Q_{1,1} := (-1)^{1+1}\cdot(S_{2,2}\cdot S_{3,3} - S_{2,3}\cdot S_{3,2}) \qquad Q_{2,1} := (-1)^{2+1}\cdot(S_{1,2}\cdot S_{3,3} - S_{1,3}\cdot S_{3,2})$$

$$Q_{1,2} := (-1)^{1+2}\cdot(S_{2,1}\cdot S_{3,3} - S_{2,3}\cdot S_{3,1}) \qquad Q_{2,2} := (-1)^{2+2}\cdot(S_{1,1}\cdot S_{3,3} - S_{1,3}\cdot S_{3,1})$$

$$Q_{1,3} := (-1)^{1+3}\cdot(S_{2,1}\cdot S_{3,2} - S_{2,2}\cdot S_{3,1}) \qquad Q_{2,3} := (-1)^{2+3}\cdot(S_{1,1}\cdot S_{3,2} - S_{1,2}\cdot S_{3,1})$$

$$Q_{3,1} := (-1)^{3+1}\cdot(S_{1,2}\cdot S_{2,3} - S_{1,3}\cdot S_{2,2})$$

$$Q_{3,2} := (-1)^{3+2}\cdot(S_{1,1}\cdot S_{2,3} - S_{2,1}\cdot S_{1,3})$$

$$Q_{3,3} := (-1)^{3+3}\cdot(S_{1,1}\cdot S_{2,2} - S_{1,2}\cdot S_{2,1})$$

$$Q = \begin{pmatrix} 2.762\times10^{8} + 1.17i\times10^{8} & -2.762\times10^{8} - 1.17i\times10^{8} & 5.525\times10^{8} + 2.34i\times10^{8} \\ -2.762\times10^{8} - 1.17i\times10^{8} & 2.762\times10^{8} + 1.17i\times10^{8} & -5.525\times10^{8} - 2.34i\times10^{8} \\ 5.525\times10^{8} + 2.34i\times10^{8} & -5.525\times10^{8} - 2.34i\times10^{8} & 1.105\times10^{9} + 4.68i\times10^{8} \end{pmatrix}$$

$$\frac{Q^{\langle 1\rangle}}{Q_{1,1}} = \begin{pmatrix} 1 \\ -1 - 5.054i\times10^{-8} \\ 2 + 1.444i\times10^{-7} \end{pmatrix} \qquad \frac{Q^{\langle 2\rangle}}{Q_{1,2}} = \begin{pmatrix} 1 \\ -1 + 1.444i\times10^{-8} \\ 2 + 1.444i\times10^{-8} \end{pmatrix}$$

$$\frac{Q^{\langle 3\rangle}}{Q_{1,3}} = \begin{pmatrix} 1 \\ -1 + 1.444i\times10^{-8} \\ 2 - 8.303i\times10^{-8} \end{pmatrix}$$

Comparison with the classical normal modes:

$$x_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$\mu_1 := x_1^T \cdot M \cdot x_1$ $\qquad \mu_1 = 8.892$

$\mu_2 := x_2^T \cdot M \cdot x_2$ $\qquad \mu_2 = 3.618$

$\mu_3 := x_3^T \cdot M \cdot x_3$ $\qquad \mu_3 = 9$

$X^{\langle 1 \rangle} := x_1 \qquad X^{\langle 2 \rangle} := x_2 \qquad X^{\langle 3 \rangle} := x_3$

$$\omega_0 := \frac{\omega_0}{100} \qquad \omega_0 = \begin{pmatrix} 0.652 \\ 1.656 \\ 2 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & 1 & 1 \\ 1.383 & -0.161 & -1 \\ 1.074 & -1.241 & 2 \end{pmatrix} \qquad \zeta = \begin{pmatrix} 0.033 \\ 0.083 \\ 0.1 \end{pmatrix}$$

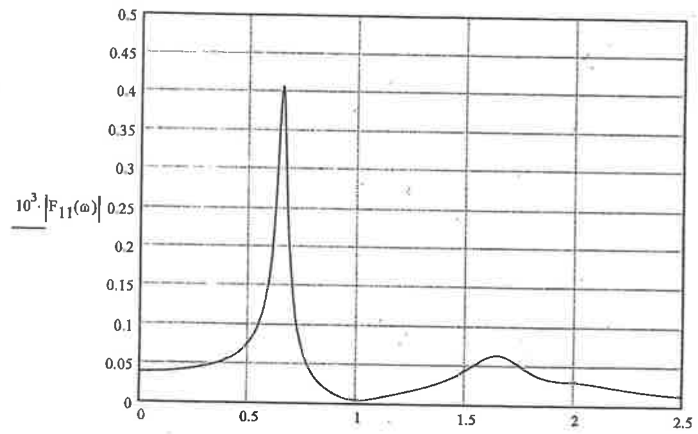
$$\mu = \begin{pmatrix} 8.892 \\ 3.618 \\ 9 \end{pmatrix}$$

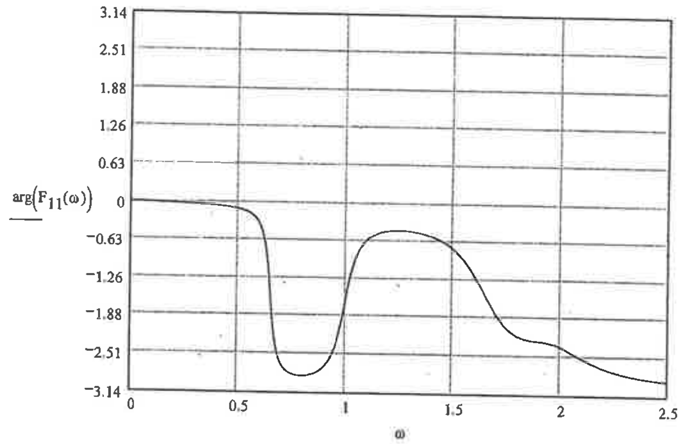
Frequency response function:

$$F(\omega) := 10^{-4} \cdot \sum_{k=1}^{3} \frac{X^{\langle k \rangle} \cdot \left(X^{\langle k \rangle}\right)^T}{\left[\left(\omega_{0_k}\right)^2 - \omega^2 + i \cdot 2 \cdot \zeta_k \cdot \omega \cdot \omega_{0_k}\right] \cdot \mu_k}$$

$$s := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad F_{11}(\omega) := s^T \cdot F(\omega) \cdot s$$

$\omega := 0, 0.00001 .. 2.5$





#

MECHANICAL VIBRATIONS

Exercise 4: Damped vibrations

Problem solutions.

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1. We assume classical normal modes:

$$\bar{X}_1 \quad \bar{X}_2 \quad \dots \quad \bar{X}_n$$

$$s_1 \quad s_2 \quad \quad s_n$$

where

$$s_i = -\zeta_i \omega_{0i} \pm \overset{\sqrt{-1}}{\overset{\parallel}{i}}\,\omega_{di}$$

$$\omega_{di} = \omega_{0i}\sqrt{1-\zeta^2} \qquad 0 \le \zeta_i < 1$$

The equations of motion:

$$\underline{M}\ddot{\bar{q}} + \underline{C}\dot{\bar{q}} + \underline{K}\bar{q} = \bar{P}$$

$$\bar{P} = \bar{P}_0 \sin\omega t$$

A particulate solution is given by

$$\bar{q}_p = \bar{q}_p(t) = \mathrm{Im}\left(\underline{F}(\omega)\,\bar{P}_0\, e^{i\omega t}\right) \equiv \bar{f}(t)$$

where

$$\underline{F}(\omega) = \sum_{i=1}^{n} \frac{\bar{X}_i \bar{X}_i^T}{(\omega_{0i}^2 - \omega^2 + i2\zeta_i\omega_{0i}\omega)\mu_i}$$

is the so-called frequency-response function and $\mu_i = \bar{X}_i^T \underline{M}\bar{X}_i$ is the modal mass.

The general solution may the be written:

$$\bar{q} = \bar{q}_h + \bar{q}_p = \bar{q}_h + \bar{f}(t)$$

where $\bar{q}_h$ is the solution to the homogeneous equation:

$$\underline{M}\ddot{\bar{q}} + \underline{C}\dot{\bar{q}} + k\bar{q} = \bar{0}$$

(the free vibrations). We know that

$$\bar{q}_h = \bar{q}_h(t) = \sum_{i=1}^{n} e^{-\xi_i \omega_{0i} t}(a_i \cos\omega_{di} t + b_i \sin\omega_{di} t)\bar{X}_i$$

where a_i, b_i $i = 1,\dots,n$ are constants to be determined by the <u>initial conditions:</u>

$$\bar{q}(0) = \bar{q}_0 \Rightarrow \bar{q}_h(0) + \bar{f}(0) = \bar{q}_0 \Rightarrow$$

$$\sum_{i=1}^{n} a_i \bar{X}_i + \bar{f}(0) = \bar{q}_0 \Rightarrow$$

$$\sum_{i=1}^{n} a_i \bar{X}_i = \bar{q}_0 - \bar{f}(0)$$

Premultiply this with $\frac{1}{\mu_j}\bar{X}_j^T \bar{M}$, then it follows that

$$a_j = \frac{1}{\mu_j}\bar{X}_j^T \underline{M}(\bar{q}_0 - \bar{f}(0))$$

Taking the time-derivative of $\bar{q} = \bar{q}(t)$ we get

$$\dot{\bar{q}}(t) = \sum_{i=1}^{n} (-\xi_i \omega_{0i}) e^{-\xi_i \omega_{0i} t}(a_i \cos\omega_{di} t + b_i \sin\omega_{di} t)\bar{X}_i + e^{-\xi_i \omega_{0i} t}(-a_i \omega_{di} \sin\omega_{di} t + b_i \omega_{di} (\cos\omega_{di} t)\bar{X}_i + \dot{\bar{f}}(t)$$

But $\dot{\bar{f}}(t) = \frac{d}{dt} Im(\underline{E}(\omega)\bar{P}_0 e^{i\omega t}) = Im(i\omega \underline{E}(\omega)\bar{P}_0 e^{i\omega t})$

Then

$$\dot{\bar{q}}(0) = \dot{\bar{q}}_0 \Rightarrow \sum_{i=1}^{n}(-\xi_i \omega_{0i} a_i + b_i \omega_{di})\bar{X}_i + \dot{\bar{f}}(0) = \dot{\bar{q}}_0 \Rightarrow$$

$$b_i = \frac{1}{\omega_{di}}\left(\frac{\bar{X}_i^T \underline{M}}{\mu_i}(\dot{\bar{q}}_0 - \dot{\bar{f}}(0)) + \frac{\xi_i \omega_{0i}}{\mu_i}\bar{X}_i^T \underline{M}(\bar{q}_0 - \bar{f}(0))\right)$$

and we have obtained <u>the solution:</u>

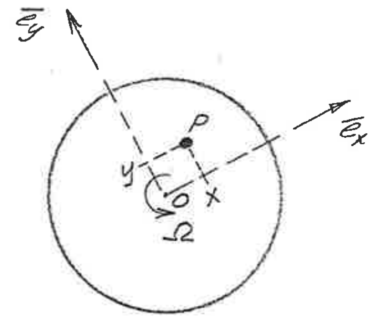
$$\bar{q}(t) = \sum_{i=1}^{n} e^{-\xi_i \omega_{0i} t}\left(\cos\omega_{di} t + \xi_i \frac{\omega_{0i}}{\omega_{di}} \sin\omega_{di} t\right) \cdot \frac{\bar{X}_i \bar{X}_i^T \underline{M}}{\mu_i}(\bar{q}_0 - \bar{f}(0)) +$$

$$\left(\sum_{i=1}^{n} e^{-\xi_i\omega_{0i}t}\,\frac{1}{\omega_{di}}\sin\omega_{di}t\;\frac{\bar{X}_i\bar{X}_i^T\underline{M}}{\mu_i}\right)(\dot{\bar{q}}_0-\dot{\bar{f}}(0))+$$

$$\bar{f}(t)$$

---

2. Position vector of the particle:



$$\bar{r} = \bar{r}_{OP} = \bar{e}_x x + \bar{e}_y y$$

Particle velocity:

$$\bar{v} = \dot{\bar{r}} = \dot{\bar{e}}_x x + \bar{e}_x\dot{x} + \dot{\bar{e}}_y y + \bar{e}_y\dot{y}$$

$$\dot{\bar{e}}_x = \bar{e}_z\Omega\times\bar{e}_x = \bar{e}_y\Omega\ ,\quad \dot{\bar{e}}_y = \bar{e}_z\Omega\times\bar{e}_y = \bar{e}_x(-\Omega)$$

and then

$$\bar{v} = \bar{e}_x(\dot{x}-\Omega y) + \bar{e}_y(\dot{y}+\Omega x)$$

a) The kinetic energy of the particle:

$$T = \frac{1}{2}m\bar{v}^2 = \frac{1}{2}m((\dot{x}-\Omega y)^2 + (\dot{y}+\Omega x)^2) =$$

$$\underbrace{\frac{m}{2}\Omega^2(x^2+y^2)}_{=T_0} + \underbrace{m\Omega(x\dot{y}-y\dot{x})}_{=T_1} + \underbrace{\frac{m}{2}(\dot{x}^2+\dot{y}^2)}_{T_2}$$

The potential energy:

$$V = V_0 + \frac{1}{2}m\omega_0^2(x^2+y^2)$$

The Lagrangian:

$$L = T - V = \frac{m}{2}(\dot{x}^2+\dot{y}^2) + m\Omega(x\dot{y}-y\dot{x}) -$$

$$\underbrace{\left\{\frac{m}{2}(\omega_0^2-\Omega^2)(x^2+y^2) + V_0\right\}}_{=V^* = V - T_0}$$ (efficient potential)

$$\frac{\partial L}{\partial\dot{x}} = m\dot{x} - m\Omega y\ ,\quad \frac{\partial L}{\partial x} = m\Omega\dot{y} - m(\omega_0^2-\Omega^2)x$$

$$\frac{\partial L}{\partial\dot{y}} = m\dot{y} + m\Omega x\ ,\quad \frac{\partial L}{\partial y} = -m\Omega\dot{x} - m(\omega_0^2-\Omega^2)y$$

Lagrange's equations (Ω = constant)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) - \frac{\partial L}{\partial x} = m\ddot{x} - m\Omega\dot{y} - m\Omega\dot{y} + m(\omega_0^2-\Omega^2)x = 0$$

or

$$m\ddot{x} - 2m\Omega\dot{y} + m(\omega_0^2 - \Omega^2)x = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = m\ddot{y} + m\Omega\dot{x} + m\Omega\dot{x} + m(\omega_0^2 - \Omega^2)y = 0$$

or

$$m\ddot{y} + 2m\Omega\dot{x} + m(\omega_0^2 - \Omega^2)y = 0$$

<u>The equations of motion</u> in matrix-format:

$$\underset{\underline{M}}{\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}} \underset{\ddot{\bar{q}}}{\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}} + \underset{\underline{G}}{\begin{pmatrix} 0 & -2m\Omega \\ 2m\Omega & 0 \end{pmatrix}} \underset{\dot{\bar{q}}}{\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}} +$$

$$\underset{\underline{K}}{\begin{pmatrix} m(\omega_0^2 - \Omega^2) & 0 \\ 0 & m(\omega_0^2 - \Omega^2) \end{pmatrix}} \underset{\bar{q}}{\begin{pmatrix} x \\ y \end{pmatrix}} = \underset{\bar{0}}{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}$$

b) It is obvious from the equations of motion that if $\omega_0^2 \neq \Omega^2$ then $\bar{q} = \bar{q}^0$ is an equilibrium state for the particle if and only if $\bar{q}^0 = \bar{0}$, i.e. $x = y = 0$. Note that if $\omega_0^2 = \Omega^2$ then <u>any</u> position is an equilibrium state.

c) $V^* = V - T_0 = \frac{m}{2}(\omega_0^2 - \Omega^2)(x^2 + y^2) + V_0$

$$\frac{\partial V^*}{\partial x} = \frac{m}{2}(\omega_0^2 - \Omega^2)2x = 0 \iff x = 0$$

$$\frac{\partial V^*}{\partial y} = \frac{m}{2}(\omega_0^2 - \Omega^2)2y = 0 \iff y = 0$$

since $\omega_0^2 \neq \Omega^2$.

$$\frac{\partial^2 V^*}{\partial x^2} = m(\omega_0^2 - \Omega^2)$$

$$\frac{\partial^2 V^*}{\partial y^2} = m(\omega_0^2 - \Omega^2)$$

$$\frac{\partial^2 V^*}{\partial x \partial y} = 0$$

From this it follows that V^* has a minimum at $x = y = 0$ if $\Omega^2 < \omega_0^2$ and a maximum if $\Omega^2 > \omega_0^2$. If $\Omega^2 = \omega_0^2$ the equilibrium is indifferent.

d) The impedance matrix for the system:

$$\underline{Z}(s) = \underline{M}s^2 + \underline{B}s + \underline{K} =$$

$$\begin{pmatrix} ms^2 + m(\omega_0^2 - \Omega^2) & -2m\Omega s \\ 2m\Omega s & ms^2 + m(\omega_0^2 - \Omega^2) \end{pmatrix}$$

The characteristic polynomial:

$$p(s) = \det \underline{Z}(s) = (ms^2 + m(\omega_0^2 - \Omega^2))^2 + 4m^2\Omega^2 s^2$$

The characteristic roots (eigenvalues): $p(s) = 0$, i.e.

$$s^4 + 2(\omega_0^2 + \Omega^2)s^2 + (\omega_0^2 - \Omega^2)^2 = 0$$

$$s^2 = -(\omega_0^2 + \Omega^2) \pm \sqrt{(\omega_0^2 + \Omega^2)^2 - (\omega_0^2 - \Omega^2)^2}$$

$$= -(\omega_0^2 + \Omega^2 \pm 2\omega_0\Omega) = -(\omega_0 \pm \Omega)^2$$

$$\Rightarrow \quad \underline{s_{1,2,3,4} = \pm i(\omega_0 \pm \Omega)}$$

The roots are thus purely imaginary and this implies Dirichlet stability. The system will oscillate around the equilibrium state irrespective of the character of the effective potential energy.

e) The viscous frictional force is represented by the damping matrix:

$$\underline{C} = \begin{pmatrix} 2m\omega_0\zeta & 0 \\ 0 & 2m\omega_0\zeta \end{pmatrix}$$

The impedance matrix: $\underline{Z}(s) =$

$$\begin{pmatrix} ms^2 + 2m\omega_0\zeta s + \Delta & -2m\Omega s \\ 2m\Omega s & ms^2 + 2m\omega_0\zeta s + \Delta \end{pmatrix}$$

where $\Delta = m(\omega_0^2 - \Omega^2)$.

The characteristic polynomial:

$p(s) = (ms^2 + 2m\omega_0\zeta s + \Delta)^2 + 4m^2\Omega^2 s^2 =$

$(ms^2 + 2m\omega_0\zeta s + \Delta)^2 - (i2m\Omega s)^2 =$

$(ms^2 + 2m(\omega_0\zeta + i\Omega)s + \Delta)\cdot(ms^2 +$

$2m(\omega_0\zeta - i\Omega)s + \Delta)$

$p(s) = 0 \iff$

$$\left| \begin{array}{l} ms^2 + 2m(\omega_0\zeta + i\Omega)s + \Delta = 0 \\ ms^2 + 2m(\omega_0\zeta - i\Omega)s + \Delta = 0 \end{array} \right.$$

Note that if s solves the first equation then s^* solves the second. The first equation gives:

$s = -\omega_0\zeta - i\Omega \pm \sqrt{\omega_0^2\zeta^2 - \Omega^2 + 2i\Omega\omega_0\zeta - \omega_0^2 + \Omega^2} =$

$-\omega_0\zeta - i\Omega \pm i\omega_0\sqrt{1 - 2i\frac{\Omega}{\omega_0}\zeta - \zeta^2} \cong$

$-\omega_0\zeta - i\Omega \pm i\omega_0(1 - i\frac{\Omega}{\omega_0}\zeta) + O(\zeta) =$

$(\pm\Omega - \omega_0)\zeta - i(\Omega \pm \omega_0) + O(\zeta)$

where $O(\zeta)$ denotes a term satisfying:

$$\lim_{\zeta \to 0} \frac{O(\zeta)}{\zeta} = 0$$

The eigenvalues

$$s_{1,2} = (\pm\Omega - \omega_0)\zeta - i(\Omega \pm \omega_0) + O(\zeta)$$

implies that we have stability (for $\zeta \ll 1$) if $|\Omega| < \omega_0$, since if $|\Omega| > \omega_0$ the solution will contain a term with exponentially growing time-dependence. Thus the introduction of damping will in this case <u>destabilize</u> the motion!

#

MECHANICAL VIBRATIONS

Exercise 5: Continuous systems

Problem solutions

1.

$$\int_0^L \varphi_n(x)\varphi_m(x)\,dx = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

Using the trig. formula $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$ gives:

$$\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx =$$

$$\frac{2}{L}\int_0^L \frac{1}{2}\left(\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right)dx =$$

$$\left[\frac{1}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L} - \frac{1}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\right]_0^L$$

$= 0$ if $n \neq m$.

If $n=m$ we get:

$$\frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \frac{1-\cos\frac{2n\pi x}{L}}{2}\,dx$$

$$\frac{2}{L}\left[\frac{1}{2}x - \frac{L}{4n\pi}\sin\frac{2n\pi x}{L}\right]_0^L = \frac{2}{L}\cdot\frac{L}{2} = 1$$

#

2. a) Take $u = u(x,t) = \hat{u}(x)\phi(t)$.

Insert this into the partial differential equation for the bar:

$$\frac{\partial}{\partial x}\left(EA\frac{\partial(\hat{u}\phi)}{\partial x}\right) = m\frac{\partial^2(\hat{u}\phi)}{\partial t^2} \Rightarrow$$

$$\left(\frac{d}{dx}\left(EA\frac{d\hat{u}}{dx}\right)\right)\phi = m\hat{u}\frac{d^2\phi}{dt^2} \Rightarrow$$

$$\frac{\frac{d}{dx}\left(EA\frac{d\hat{u}}{dx}\right)}{m\hat{u}} = \frac{\frac{d^2\phi}{dt^2}}{\phi} = -\lambda \qquad (1)$$

where λ is a constant. Note that the left membrum in (1) does not depend on t and that:

$$\frac{\frac{d^2\phi}{dt^2}}{\phi} \quad \text{does not depend on } x!$$

(1) is then equivalent to

$$-\frac{1}{m}\frac{d}{dx}\left(EA\frac{d\hat{u}}{dx}\right) = \lambda\hat{u}$$

$$\frac{d^2\phi}{dt^2} + \lambda\phi = 0$$

and we have the boundary condition:

$$\alpha_1 u(0,t) + \beta_1 \frac{\partial u(0,t)}{\partial x} = \alpha_1 \hat{u}(0)\phi(t) +$$

$$\beta_1 \frac{d\hat{u}(0)}{dx}\phi(t) = 0 \quad \forall t \quad \Rightarrow \quad \alpha_1 \hat{u}(0) +$$

$$\beta_1 \frac{d\hat{u}(0)}{dx} = 0$$

The other boundary condition, at $x = L$, is demonstrated in the same way:

$$\alpha_2 \hat{u}(L) + \beta_2 \frac{d\hat{u}}{dx}(L) = 0$$

b) $$\int_0^L (\mathcal{L}\hat{u})\hat{v}\, m\, dx = -\int_0^L \frac{1}{m}\frac{d}{dx}\left(EA\frac{d\hat{u}}{dx}\right)\hat{v}\, m dx =$$

$$-\left[EA\frac{d\hat{u}}{dx}\hat{v}\right]_0^L + \int_0^L EA\frac{d\hat{u}}{dx}\frac{d\hat{v}}{dx}dx \quad \text{and}$$

$$\int_0^L \hat{u}(\mathcal{L}\hat{v}) m dx = -\left[EA\hat{u}\frac{d\hat{v}}{dx}\right]_0^L +$$

$$\int_0^L EA\frac{d\hat{u}}{dx}\frac{d\hat{v}}{dx}dx.$$

Thus

$$\int_0^L (\mathcal{L}\hat{u})\hat{v}\, m dx - \int_0^L \hat{u}(\mathcal{L}\hat{v}) m dx =$$

$$-\left[EA\frac{d\hat{u}}{dx}\hat{v}\right]_0^L + \left[EA\hat{u}\frac{d\hat{v}}{dx}\right]_0^L =$$

$$EA\left\{\hat{u}(L)\frac{d\hat{v}}{dx}(L) - \frac{d\hat{u}}{dx}(L)\hat{v}(L)\right\} +$$

$$EA\left\{\frac{d\hat{u}}{dx}(0)\hat{v}(0) - \hat{u}(0)\frac{d\hat{v}}{dx}(0)\right\} \qquad (2)$$

But from boundary conditions:

$$\left.\begin{aligned}\alpha_2\hat{u}(L) + \beta_2\frac{d\hat{u}}{dx}(L) = 0\\ \alpha_2\hat{v}(L) + \beta_2\frac{d\hat{v}}{dx}(L) = 0\end{aligned}\right\} \Rightarrow$$

$$\alpha_2\beta_2\ \hat{u}(L)\frac{d\hat{v}}{dx}(L) - \alpha_2\beta_2\frac{d\hat{u}}{dx}(L)\hat{v}(L) = 0$$

$\Rightarrow (\alpha_2\beta_2 \neq 0) \Rightarrow$

$$\hat{u}(L)\frac{d\hat{v}}{dx}(L) - \frac{d\hat{u}}{dx}(L)\hat{v}(L) = 0 \qquad (3)$$

In the same way we get:

$$\frac{d\hat{u}}{dx}(0)\hat{v}(0) - \hat{u}(0)\frac{d\hat{v}}{dx}(0) = 0 \qquad (4)$$

Using (2)-(4) we get

$$\int_0^L(\mathcal{L}\hat{u})\hat{v}\,m\,dx - \int_0^L\hat{u}(\mathcal{L}\hat{v})m\,dx = 0$$

and $\mathcal{L}$ i a <u>symmetric</u> differential operator.

c) $\begin{vmatrix}\mathcal{L}\hat{u}_1 = \lambda_1\hat{u}_1\\ \mathcal{L}\hat{u}_2 = \lambda_2\hat{u}_2\end{vmatrix}$

$\hat{u}_1$ and $\hat{u}_2$ satisfying the boundary conditions. $\mathcal{L}$ is symmetric $\Rightarrow$

$$\int_0^L(\mathcal{L}\hat{u}_1)\hat{u}_2\,m\,dx = \int_0^L\hat{u}_1(\mathcal{L}\hat{u}_2)m\,dx \Rightarrow$$

$$\lambda_1\int_0^L\hat{u}_1\hat{u}_2\,m\,dx = \lambda_2\int_0^L\hat{u}_1\hat{u}_2\,m\,dx \Rightarrow$$

$$(\lambda_1-\lambda_2)\int_0^L\hat{u}_1\hat{u}_2\,m\,dx = 0$$

Thus

$$\lambda_1 \neq \lambda_2 \Rightarrow \int_0^L\hat{u}_1\hat{u}_2\,m\,dx$$

i.e the functions $\hat{u}_1 = \hat{u}_1(x)$ and $\hat{u}_2 = \hat{u}_2(x)$ are <u>orthogonal</u> with respect to the scalar product

$$\langle f,g \rangle = \int_0^L f(x)g(x)\,m(x)\,dx$$

#

3. The natural frequencies for the string with fixed ends is:

$$\omega_n = n\pi\sqrt{\frac{T_0}{m_0 L^2}} \qquad (1)$$

where $m_0 L$ is the total mass of the string; $m_0 L = 110 \cdot 10^{-3}\,kg$.

$L = 1.4\,m$ and the first natural frequency:

$$\omega_1 = 2\pi f_1 = 2\pi \cdot 424\ s^{-1}$$

From (1) we get

$$\underline{T_0} = m_0 L^2 \left(\frac{\omega_1}{\pi}\right)^2 = 110 \cdot 10^{-3}\,kg \cdot 1.4\,m =$$

$$\left(\frac{2\pi \cdot 424 s^{-1}}{\pi}\right)^2 \cong \underline{111\ kN}$$

#

4. Mode shapes for the vibrating string are given by the eq.

$$\frac{d^2\hat{w}}{dx^2} + \Omega^2\hat{w} = 0 \qquad (1)$$

where

$$\Omega^2 = \frac{\omega^2 m_0}{T_0} = \frac{\omega^2}{c^2}$$

Solutions to (1):

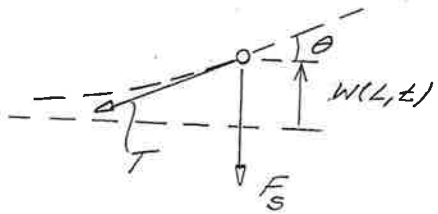
$$\hat{w} = \hat{w}(x) = a\sin\Omega x + b\cos\Omega x$$

Boundary conditions:

$$\hat{w}(0) = 0 \quad \Rightarrow \quad b = 0$$

$$T_0\,\frac{d\hat{w}}{dx}(L) = -k\,\hat{w}(L) \qquad (2)$$

The second condition follows from



the equilibrium equation for the right hand of the string (see fig. 1)

$$(\uparrow): \; -T\sin\theta - F_s = 0 \qquad (3)$$

where T is the string tension and F_s the spring force. Now

$$\left.\begin{aligned} T &= T_0\sqrt{1+\left(\frac{d\hat{w}}{dx}\right)^2} \\ \sin\theta &= \frac{d\hat{w}}{dx}\left(\sqrt{1+\left(\frac{d\hat{w}}{dx}\right)^2}\right)^{-1} \end{aligned}\right\} \Rightarrow$$

$$\left|\begin{aligned} T\sin\theta &= T_0\frac{d\hat{w}}{dx} \\ F_s &= k\hat{w}(L) \end{aligned}\right. \qquad (4)$$

Now (3) & (4) $\Rightarrow$ (2) which implies

$$T_0\,\Omega a\cos\Omega L = -ka\sin\Omega L \Rightarrow$$

$$\tan\Omega L = -\frac{T_0\,\Omega L}{kL}$$

or

$$\left|\; \tan z = -\frac{T_0}{kL}z, \quad (z=\Omega L)\right.$$

$T_0 = 10000\,N$

$k = 8000\,Nm^{-1}$

$L = 1.0\,m$

$$\left|\; \tan z = -12.5\,z\right.$$

Roots to this equation are given in the appended Matcad document. We read off:

$$\left|\; z_1 = 1.62, \quad z_2 = 4.712, \quad z_3 = 7.854\right.$$

For large n we have

$$z_n \cong (2n-1)\frac{\pi}{2L}$$

See figure in Matcad document!

$$\Omega_n = \frac{z_n}{L} = z_n$$

and the mode shapes are given by

$$\hat{W}(x) = a \sin \Omega_n t$$

Corresponding natural frequencies:

$$\omega_n = \Omega_n \sqrt{\frac{T_0}{m_0}}$$

where $m_0 = \rho A_0 = 0.025\ kgm^{-1}$ and we get:

$$\omega_1 \cong 3.2 \cdot 10^3\ s^{-1}, \quad \omega_2 \cong 9.4 \cdot 10^3\ s^{-1}$$

$$\omega_3 \cong 15.7 \cdot 10^3\ s^{-1}$$

Se Matcad document!

Exercise 5:4 (Mathcad document)

INPUT:

$T_0 := 10 \cdot 10^4$

$k := 8000$

$\rho := 8000$

$L := 1.0$

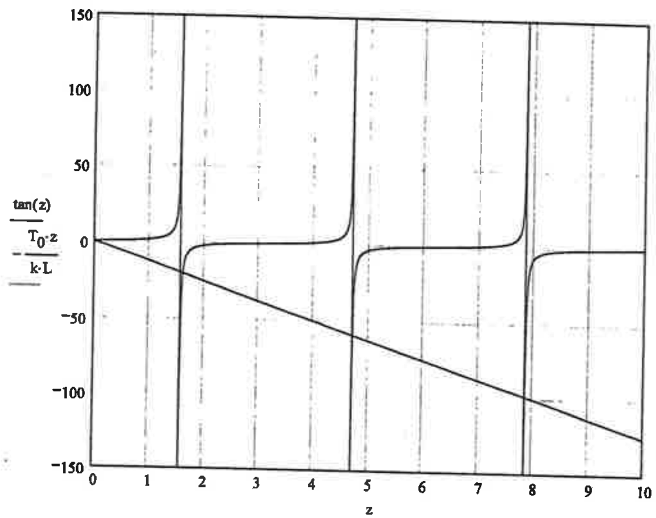
$r := 1.0 \cdot 10^{-3}$

CALCULATIONS:

$A := \pi \cdot r^2$

$m_0 := \rho \cdot A \qquad m_0 = 0.025$

$z := 0, 0.001 .. 10$



13

$z := 2$

Given

$$\tan(z) + \frac{T_0 \cdot z}{k \cdot L} = 0$$

$z := \text{Find}(z)$

$z = 1.62$ $\qquad \frac{\pi}{2 \cdot L} = 1.571$ $\qquad \omega := \sqrt{\frac{T_0 \cdot z^2}{m_0}}$ $\qquad \omega = 3.232 \times 10^3$

$z := 5$

Given

$$\tan(z) + \frac{T_0 \cdot z}{k \cdot L} = 0 \qquad T_0 = 1 \times 10^5$$

$z := \text{Find}(z)$

$z = 4.729$ $\qquad 3 \cdot \frac{\pi}{2 \cdot L} = 4.712$ $\qquad \omega := \sqrt{\frac{T_0 \cdot z^2}{m_0}}$ $\qquad \omega = 9.434 \times 10^3$

$z := 8$

Given

$$\tan(z) + \frac{T_0 \cdot z}{k \cdot L} = 0 \qquad \frac{T_0}{k \cdot L} = 12.5$$

$z := \text{Find}(z)$

$z = 7.864$ $\qquad 5 \cdot \frac{\pi}{2 \cdot L} = 7.854$ $\qquad \omega := \sqrt{\frac{T_0 \cdot z^2}{m_0}}$ $\qquad \omega = 1.569 \times 10^4$

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14

5. Differential equation for mode-shapes (longitudinal vibrations):

$$EA \frac{d^2 \hat{u}}{dx^2} + \omega^2 m_0 \hat{u} = 0$$

where

$$m_0 = \frac{m}{L_0}$$

Solutions:

$$\hat{u} = \hat{u}(x) = a \sin \lambda \frac{x}{L_0} + b \cos \lambda \frac{x}{L_0}$$

$$\lambda^2 = \frac{\omega^2 m L_0}{EA_0} \qquad (1)$$

Boundary conditions ("clamped-clamped"):

$$u(0,t) = u(L,t) = 0 \quad \Rightarrow$$

$$\hat{u}(0) = \hat{u}(L) = 0 \quad \Rightarrow \quad b = 0,$$

$$a \sin \lambda \frac{L_0}{L_0} = a \sin \lambda = 0 .$$

$a \neq 0 \Rightarrow \sin\lambda = 0 \Leftrightarrow \lambda_n = n\pi$,

$n = 1, 2, \ldots$

From this, and (1), we find the natural frequencies:

$$\omega_n = \lambda_n \sqrt{\frac{EA_0}{mL_0}} = n\pi \sqrt{\frac{EA_0}{mL_0}}$$

#

6. Let $u = u(x,t)$ denote this displacement of the bar. Then

$$EA \frac{\partial^2 u}{\partial x^2} = \ddot{u} m \qquad (1)$$

with the boundary conditions:

$$u(0,t) = 0$$

$$N(L,t) = P(t) \Leftrightarrow \frac{\partial u}{\partial x}(L,t) = \frac{1}{EA} P(t) \qquad t \geqq 0$$

and the initial conditions

$$u(x,0) = 0, \quad \dot{u}(x,0) = 0$$

$$0 \leqq x \leqq L$$

Let $u_q = u_q(x,t)$ denote the so-called "quasi-static solution, i.e.

$$EA \frac{\partial^2 u_q}{\partial x^2} = \ddot{u}_q m = 0$$

$$u_q(0,t) = 0, \quad \frac{\partial u_q}{\partial x}(L,t) = \frac{P(t)}{EA}$$

which implies

$$u_q(x,t) = \frac{P(t)}{EA} x$$

The difference $u_h(x,t) = u(x,t) - u_q(x,t)$ will then satisfy:

$$EA \frac{\partial^2 u_h}{\partial x^2} = \ddot{u}_h m$$

$$u_h(0,t) = 0, \quad \frac{\partial u_h}{\partial x}(L,t) = 0$$

which means that for u_h we have homogeneous boundary conditions. We may then decompose u_h in mode shape functions $\hat{u}_i = \hat{u}_i(x)$

$$u_h(x,t) = \sum_{i=1}^{\infty} \hat{u}_i(x)\,\eta_i(t) \qquad (2)$$

where $\hat{u}_i$ satisfy:

$$EA\frac{d^2\hat{u}_i}{dx^2} + \omega^2 m\hat{u}_i = 0$$

$$\hat{u}_i(0) = 0, \quad \frac{d\hat{u}_i}{dx}(L) = 0.$$

which means that

$$\hat{u}_i(x) = \sqrt{\frac{2}{mL}}\,\sin\,(2i-1)\frac{\pi x}{2L} \qquad i = 1, 2, \ldots$$

and

$$\omega_i = (2i-1)\frac{\pi}{2}\sqrt{\frac{EA}{mL^2}}$$

The solution $u = u(x,t)$ will then be on the form:

$$u(x,t) = \frac{P(t)}{EA}x + \sum_{i=1}^{\infty}\eta_i(t)\sqrt{\frac{2}{mL}}\,\sin(2i-1)\frac{\pi x}{2L}$$

We now insert this expression into the diff. eq.(1) in order to determine the functions $\eta_i = \eta_i(t)$.

$$EA\frac{\partial^2 u}{\partial x^2} = \sum_{i=1}^{\infty}\eta_i EA\frac{d^2\hat{u}_i}{dx^2} =$$

$$-\sum_{i=1}^{\infty}\eta_i\omega_i^2 m\hat{u}_i = \ddot{u}m = (\frac{\ddot{P}(t)}{EA}x +$$

$$\sum_{i=1}^{\infty}\ddot{\eta}_i\hat{u}_i)m$$

Mutiplying this equation with $\hat{u}_k$ and then integrating gives

$$-\sum_{i=1}^{\infty}\eta_i\omega_i^2\int_0^L\hat{u}_k\hat{u}_i m\,dx = \int_0^L\hat{u}_k\frac{\ddot{P}(t)}{EA}x m\,dx$$

$$+\sum_{i=1}^{\infty}\ddot{\eta}_i\int_0^L\hat{u}_k\hat{u}_i m\,dx \qquad (3)$$

We now use the orthonormality

conditions:

$$\int_0^L \hat{u}_k \hat{u}_i m\,dx = \delta_{ik}$$

and then from (3):

$$\ddot{\eta}_k + \omega_i^2 \eta_k = -\int_0^L \hat{u}_k \frac{\ddot{p}(t)}{EA} x\, m\,dx$$

But in this case $\ddot{p}(t) = 0$ and then

$$\ddot{\eta}_k + \omega_i^2 \eta_k = 0$$

with the solution

$$\eta_k(t) = \eta_k(0)\cos\omega_k t + \dot{\eta}_k(0)\frac{\sin\omega_k t}{\omega_k}$$

From (2) it follows that:

$$u_h(x,0) = \sum_{i=1}^{\infty} \hat{u}_i(x)\eta_i(0)$$

$$\dot{u}_h(x,0) = \sum_{i=1}^{\infty} \hat{u}_i(x)\dot{\eta}_i(0)$$

and then using ortonormality:

$$\eta_k(0) = \int_0^L \hat{u}_k(x)\, u_h(x,0)\, m\,dx$$

$$\dot{\eta}_k(0) = \int_0^L \hat{u}_k(x)\, \dot{u}_h(x,0)\, m\,dx$$

But, using the initial conditions:

$$u_h(x,0) = u(x,0) - u_q(x,0) = 0 - \frac{P_0}{EA}x$$

$$\dot{u}_h(x,0) = u(x,0) - u_q(x,0) = 0$$

we obtain:

$$\eta_k(0) = \int_0^L \hat{u}_k(x)\left(-\frac{P_0}{EA}x\right) m\,dx$$

$$\dot{\eta}_k(0) = 0$$

The solution may then be written:

$$u(x,t) = \frac{P_o x}{EA} - \sum_{i=1}^{\infty} \frac{2}{mL} \left(\int_0^L \frac{P_o x}{EA} \sin\left((2i-1)\frac{\pi x}{2L}\right) dx \right)$$

$$\sin\left((2i-1)\frac{\pi x}{2L}\right) \cos \omega_i t$$

After calculation of the integral we get:

$$u(x,t) = \frac{P_o}{EA} x - \frac{8 P_o L}{\pi^2 EA} \sum_{i=1}^{\infty} \left(\frac{(-1)^{i-1}}{(2i-1)^2} \sin\left((2i-1)\frac{\pi x}{2L}\right) \right.$$

$$\left. \cos\left((2i-1)\frac{\pi c}{2L} t\right) \right)$$

where $c = \sqrt{\frac{EA}{m}}$ is the wave-speed.

The calculation of $u(\frac{L}{2}, t)$ is performed in the following Mathcad document.

Exercise 5:6 (Mathcad document)

INPUT:

$E := 200 \cdot 10^9$

$\rho := 8000$

$L := 5.0$

$A := 2 \cdot 10^{-4}$

$P_0 := 100000$

CALCULATIONS:

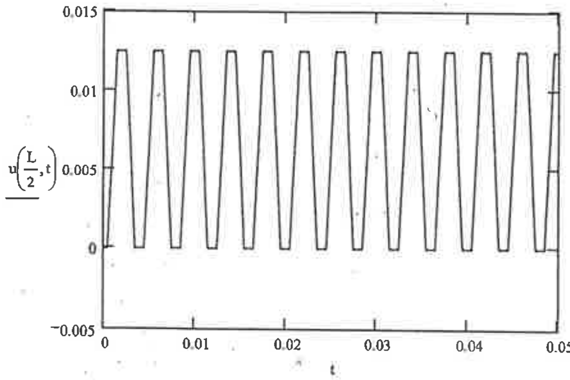
$m := \rho \cdot A$ $\quad m = 1.6$

$c := \sqrt{\frac{E \cdot A}{m}}$ $\quad c = 5 \times 10^3$

$n := 10$

$$u(x,t) := \frac{P_0}{E \cdot A} \cdot x - \frac{8 P_0 \cdot L}{\pi^2 \cdot E \cdot A} \cdot \sum_{i=1}^{n} \frac{(-1)^{(i-1)}}{(2 \cdot i - 1)^2} \cdot \sin\left[\frac{(2 \cdot i - 1) \cdot \pi \cdot x}{2 \cdot L}\right] \cdot \cos\left[(2 \cdot i - 1) \cdot \frac{\pi \cdot c}{2 \cdot L} \cdot t\right]$$

$t := 0, 0.00001 .. 0.05$



7. Modeshapes $\hat{w} = \hat{w}(x)$ for the transverse bending of a beam are determined by the differential equation:

$$\frac{d^4\hat{w}}{dx^4} + \mu^4 \hat{w} = 0 \qquad (1)$$

$$\mu^4 = \frac{\omega^2 m}{EI}, \quad m = \rho A \qquad (2)$$

The general solution to (1) is given by:

$$\hat{w}(x) = a\sin\mu x + b\cos\mu x + c\sinh\mu x + d\cosh\mu x \qquad (3)$$

Boundary conditions:

$$\hat{w}(0) = 0, \quad \frac{d\hat{w}}{dx}(0) = 0$$

$$\hat{w}(L) = 0, \quad \frac{d^2\hat{w}}{dx^2}(L) = 0 \quad (M(L,t)=0)$$

From these we get the conditions:

$$\hat{w}(0) = b + d = 0 \Rightarrow d = -b$$

$$\frac{d\hat{w}}{dx}(0) = \mu(a+c) = 0 \Rightarrow c = -a$$

$$\hat{w}(L) = a\sin\mu L + b\cos\mu L + c\sinh\mu L + d\cosh\mu L = 0$$

$$\frac{d^2\hat{w}}{dx^2}(L) = -a\mu^2\sin\mu L - b\mu^2\cos\mu L + c\mu^2\sinh\mu L + d\mu^2\cosh\mu L = 0$$

This implies the following two conditions:

$$a(\sin\mu L - \sinh\mu L) + b(\cos\mu L - \cosh\mu L) = 0$$
$$a(\sin\mu L + \sinh\mu L) + b(\cos\mu L + \cosh\mu L) = 0 \qquad (4)$$

$(a,b) \neq (0,0) \Rightarrow$

$$(\sin\mu L - \sinh\mu L)(\cos\mu L + \cosh\mu L) - (\cos\mu L - \cosh\mu L)(\sin\mu L + \sinh\mu L) = 0$$

$\Rightarrow$

$$\tan\mu L = \tanh\mu L$$

Putting $\mu L = z$ we have the equation:

$$\tan z = \tanh z$$

with the first three solutions:

$$z_1 \cong 3.93 \;, \quad z_2 \cong 7.07 \;, \quad z_3 \cong 10.21$$

See appended Mathcad document.

From (2) we get:

$$\omega = \mu^2 \sqrt{\frac{EI}{m}} = \left(\frac{z}{L}\right)\sqrt{\frac{EI}{m}}$$

which gives

$$\omega_1 = 3.28 \cdot 10^3, \quad \omega_2 = 10.62 \cdot 10^3$$

$$\omega_3 = 22.16 \cdot 10^3 \quad (s^{-1})$$

The mode-shapes are given by (3):

$$b = -\frac{\sin\mu_n L - \sinh\mu_n L}{\cos\mu_n L - \cosh\mu_n L}\, a$$

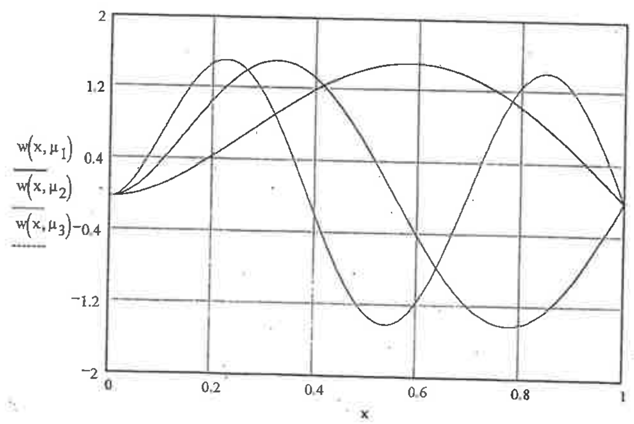
$$c = -a$$

$$d = -b = \frac{\sin\mu_n L - \sinh\mu_n L}{\cos\mu_n L - \cosh\mu_n L}$$

and then from (3)

$$\hat{w}(x) = a\Big(\sin\mu x - \frac{\sin\mu_n L - \sinh\mu_n L}{\cos\mu_n L - \cosh\mu_n L}\cos\mu x$$
$$- \sinh\mu x + \frac{\sin\mu_n L - \sinh\mu_n L}{\cos\mu_n L - \cosh\mu_n L}\cosh\mu x\Big)$$

where a is an arbitrary real constant. The three first modeshapes are plotted in the figure below.



Exercise 5:7 (Mathcad document)

INPUT:

$\rho := 8000$

$E := 200\cdot 10^{9}$

$L := 1.0$

$A := 2.6\cdot 10^{-3}$

$I := 4.7\cdot 10^{-6}$

CALCULATIONS:

$m := \rho\cdot A$ $\qquad m = 20.8$

Eigenfrequencies

$z := 2$

Given

$\tan(z) - \tanh(z) = 0$

$z := \mathrm{Find}(z)$

$z = 3.927$ $\qquad \dfrac{(4\cdot 1 + 1)\cdot\pi}{4} = 3.927$ $\qquad \omega := \left(\dfrac{z}{L}\right)^{2}\cdot\sqrt{\dfrac{E\cdot I}{m}}$ $\qquad \omega = 3.278\times 10^{3}$

$\mu_1 := \dfrac{z}{L}$ $\qquad \mu_1 = 3.927$

$z := 5$

Given

$\tan(z) - \tanh(z) = 0$

$z := \mathrm{Find}(z)$

$z = 7.069$ $\qquad \dfrac{(4\cdot 2 + 1)\cdot\pi}{4} = 7.069$ $\qquad \omega := \left(\dfrac{z}{L}\right)^{2}\cdot\sqrt{\dfrac{E\cdot I}{m}}$ $\qquad \omega = 1.062\times 10^{4}$

$\mu_2 := \dfrac{z}{L}$ $\qquad \mu_2 = 7.069$

$z := 8$

Given

$\tan(z) - \tanh(z) = 0$

$z := \mathrm{Find}(z)$

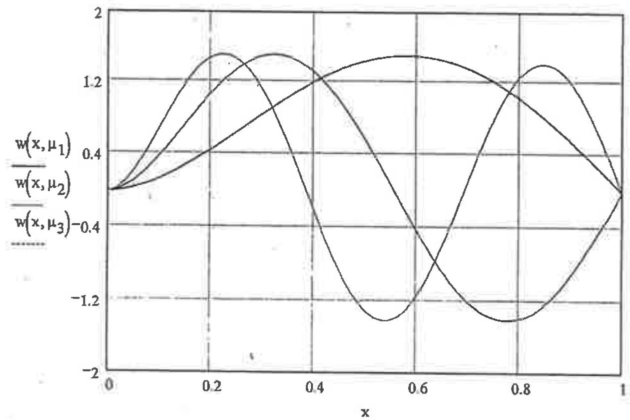
$z = 10.21$ $\qquad \dfrac{(4\cdot 3 + 1)\cdot\pi}{4} = 10.21$ $\qquad \omega := \left(\dfrac{z}{L}\right)^{2}\cdot\sqrt{\dfrac{E\cdot I}{m}}$ $\qquad \omega = 2.216\times 10^{4}$

$\mu_3 := \dfrac{z}{L}$ $\qquad \mu_3 = 10.21$

Modeshapes

$$w(x,\mu) := \sin(\mu\cdot x) - \frac{\sin(\mu\cdot L) - \sinh(\mu\cdot L)}{\cos(\mu\cdot L) - \cosh(\mu\cdot L)}\cdot\cos(\mu\cdot x) - \sinh(\mu\cdot x) + \frac{\sin(\mu\cdot L) - \sinh(\mu\cdot L)}{\cos(\mu\cdot L) - \cosh(\mu\cdot L)}\cdot\cosh(\mu\cdot x)$$

$x := 0, 0.001\cdot L .. L$



8. The free transverse vibrations of the beam are governed by the diff. eq. ($w = w(x,t)$, $0 \leq x \leq L$):

$$EI \frac{\partial^4 w}{\partial x^4} + m \frac{\partial^2 w}{\partial t^2} = 0$$

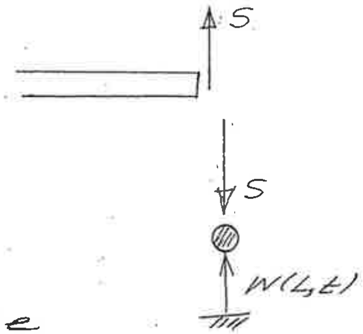
with the boundary conditions

$$w(0,t) = 0 \;, \quad \frac{\partial w}{\partial x}(0,t) = 0$$

$$M(L,t) = 0 \;, \quad -S(L,t) = Q \frac{\partial^2 w}{\partial t^2}(L,t)$$

where $Q = 10\,kg$ is the mass of the weight.

For stationary waves we have:



$$w(x,t) = \hat{w}(x) \sin \omega t$$

where the modeshape $\hat{w}$ satisfies:

$$-S = Q \frac{\partial^2 w}{\partial t^2}(L,t)$$

$$\frac{d^4 \hat{w}}{dx^4} - \mu^4 \hat{w} = 0 \qquad (1)$$

where

$$\mu^4 = \frac{\omega^2 m}{EI}$$

with the associated boundary conditions:

$$\hat{w}(0) = 0 \;, \quad \frac{d\hat{w}}{dx}(0) = 0$$

$$\frac{d^2 \hat{w}}{dx^2}(L) = 0 \;, \quad EI \frac{d^3 \hat{w}}{dx^3} = -Q\omega^2 \hat{w} \qquad (2)$$

The general solution of (1) is given by

$$\hat{w}(x) = a \sin \mu x + b \cos \mu x + c \sinh \mu x + d \cosh \mu x$$

Application of the boundary conditions (2) leads to

$$\hat{w}(0) = 0 \Rightarrow b + d = 0 \Rightarrow d = -b$$

$$\frac{d\hat{w}}{dx}(0) = 0 \Rightarrow \mu(a + c) = 0 \Rightarrow c = -a \qquad (3)$$

$$\frac{d^2\hat{W}}{dx^2}(L)=0 \Rightarrow \mu^2(-a\sin\mu L - b\cos\mu L +$$

$$c\sinh\mu L + d\cosh\mu L) = 0$$

$$\frac{d^3\hat{W}}{dx^3}(L) = -\frac{Q\omega^2}{EI}\hat{W} = -\frac{Q}{m}\mu^4\hat{W} \Rightarrow$$

$$\frac{m}{Q}\,\frac{d^3\hat{W}(L)}{dx^3} + \mu^4\hat{W} = 0$$

But $\dfrac{d^3\hat{W}(L)}{dx^3} = -a\mu^3\cos\mu x + b\mu^3\sin\mu x +$

$$c\mu^3\cosh\mu x + d\mu^3\sinh\mu x$$

and then:

$$a(\mu\sin\mu L - \frac{m}{Q}\cos\mu L) + b(\mu\cos\mu L +$$

$$\frac{m}{Q}\sin\mu L) + c(\mu\sinh\mu L + \frac{m}{Q}\cosh\mu L) +$$

$$d(\mu\cosh\mu L + \frac{m}{Q}\sinh\mu L) = 0$$

Simplifying, by using (3) we get

$$a(\sin\mu L + \sinh\mu L) + b(\cos\mu L + \cosh\mu L) = 0$$

$$a(\mu\sin\mu L - \mu\sinh\mu L - \frac{m}{Q}\cos\mu L -$$

$$\frac{m}{Q}\cosh\mu L) + b(\mu\cos\mu L - \mu\cosh\mu L +$$

$$\frac{m}{Q}\sin\mu L - \frac{m}{Q}\sinh\mu L) = 0$$

Nontrivial solutions $(a,b) \neq (0,0)$ exist and only if:

$$1 + \cos\mu L\cosh\mu L + \frac{\mu L Q}{mL}(\cos\mu L\sinh\mu L -$$

$$\sin\mu L\cosh\mu L) = 0$$

or with $\mu L = z$ and $k = \dfrac{Q}{mL}$

$$1 + \cos z\cosh z + kz(\cos z\sinh z -$$

$$\sin z\cosh z) = 0$$

The three lowest roots of this equation are (see Matcad document)

$z_1 \cong 1.429$, $z_2 \cong 4.117$, $z_3 \cong 7.194$

and the corresponding natural frequencis

$$\omega = \left(\frac{z}{L}\right)^2 \sqrt{\frac{EI}{m}}$$

$\omega_1 \cong 434.3\ s^{-1}$, $\omega_2 = 3.60 \cdot 10^3\ s^{-1}$

$\omega_3 \cong 11.0 \cdot 10^3\ s^{-1}$

Exercise 5:8 (Mathcad document)

INPUT:

$\rho := 8000$

$E := 200 \cdot 10^9$

$L := 1.0$

$A := 2.6 \cdot 10^{-3}$

$I := 4.7 \cdot 10^{-6}$

$Q := 10$

CALCULATIONS:

$m := \rho \cdot A$ $\quad m = 20.8$

$k := \frac{Q}{m \cdot L}$ $\quad k = 0.481$

Eigenfrequencies

$z := 2$

Given

$1 + \cos(z)\cdot\cosh(z) + k\cdot z\cdot(\cos(z)\cdot\sinh(z) - \sin(z)\cdot\cosh(z)) = 0$

$z := \text{Find}(z)$

$z = 1.429$ $\quad \omega := \left(\frac{z}{L}\right)^2 \cdot \sqrt{\frac{E \cdot I}{m}}$ $\quad \omega = 434.313$

$z := 4$

Given

$1 + \cos(z)\cdot\cosh(z) + k\cdot z\cdot(\cos(z)\cdot\sinh(z) - \sin(z)\cdot\cosh(z)) = 0$

$z := \text{Find}(z)$

$z = 4.117$ $\quad \omega := \left(\frac{z}{L}\right)^2 \cdot \sqrt{\frac{E \cdot I}{m}}$ $\quad \omega = 3.603 \times 10^3$

z := 7

Given

$1 + \cos(z)\cdot\cosh(z) + k\cdot z\cdot(\cos(z)\cdot\sinh(z) - \sin(z)\cdot\cosh(z)) = 0$

z := Find(z)

$z = 7.194$ $\omega := \left(\frac{z}{L}\right)^2\cdot\sqrt{\frac{E\cdot I}{m}}$ $\omega = 1.1\times 10^4$

#

9. We use the mode decomposition:

$$w(x,t) = \sum_{i=1}^{\infty} \hat{w}_i(x)\,\eta_i(t) \qquad (1)$$

where the mode shapes $\hat{w}_i$ are given by

$$\hat{w}_i(x) = \sqrt{\frac{2}{L}}\sin\frac{i\pi x}{L}, \qquad i = 1, 2, \ldots \qquad (2)$$

These modeshapes correspond to the free vibrations of the simply supported beam.

If (1) is inserted into the diff. eq.

$$EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = p, \quad m = \rho A$$

we get

$$\sum_{i=1}^{\infty} EI\frac{d^4\hat{w}_i}{dx^4}\eta_i + \sum_{i=1}^{\infty} m\hat{w}_i\ddot{\eta}_i = p$$

But

$$\frac{d^4\hat{W}_i}{dx^4} = \mu_i^4 \hat{W}_i \quad , \quad \mu_i^4 = \frac{\omega_i^2 m}{EI}$$

and then

$$\sum_{i=1}^{\infty} (\omega_i^2 \eta_i + \ddot{\eta}_i) \hat{W}_i = \frac{p}{m} = \frac{\hat{p}(x)}{m} \sin\omega t$$

Using the orthonormality of the modeshapes:

$$\int_0^L \hat{W}_i(x) \hat{W}_j(x)\, dx = \delta_{ij}$$

We find that

$$\omega_i^2 \eta_i + \ddot{\eta}_i = \left(\int_0^L \hat{W}_i(x) \frac{\hat{p}(x)}{m} dx\right) \sin\omega t \qquad (3)$$

The steady-state solution to this is given by:

$$\eta_i = \eta_i(t) = \frac{\int_0^L \hat{W}_i(x) \frac{\hat{p}(x)}{m} dx}{\omega_i^2 - \omega^2} \sin\omega t$$

Note that we neglect the homogeneous solution to (3)! We assume that this solution is damped out in the long run.

The forced steady-state response may then be written:

$$W(x,t) = \sum_{i=1}^{\infty} \hat{W}_i(x)\, \eta_i(t) =$$

$$\sum_{i=1}^{\infty} \hat{W}_i(x) \frac{\int_0^L \hat{W}_i(x) \frac{\hat{p}(x)}{m} dx}{\omega_i^2 - \omega^2} \sin\omega t =$$

$$\sum_{i=1}^{\infty} \frac{2}{mL} \int_0^L \sin\frac{i\pi x}{L} \hat{p}(x)\, dx \; \frac{\sin\frac{i\pi x}{L} \sin\omega t}{\omega_i^2 - \omega^2}$$

a) $\hat{p}(x) = p_0 \Rightarrow$

$$\int_0^L \sin\frac{i\pi x}{L} p_0\, dx = p_0 \left[-\frac{L}{i\pi} \cos\frac{i\pi x}{L}\right]_0^L =$$

$$-\frac{p_0 L}{i\pi}((-1)^i - 1) = \frac{p_0 L}{i\pi}((-1)^{i-1} + 1)$$

and then

$$W(x,t) = \sum_{i=1}^{\infty} \frac{2p_0}{i\pi m}\left((-1)^{i-1}+1\right) \frac{\sin\frac{i\pi x}{L}\sin\omega t}{\omega_i^2-\omega^2} =$$

$$\sum_{i=1}^{\infty} \frac{4p_0}{(2i-1)\pi m} \frac{\sin\frac{(2i-1)\pi x}{L}\sin\omega t}{\omega_i^2-\omega^2}$$

$$W\left(\frac{L}{2},t\right) = \sum_{i=1}^{\infty} \frac{4p_0(-1)^{i+1}}{(2i-1)\pi m} \frac{\sin\omega t}{\omega_i^2-\omega^2}$$

b) $\hat{p}(x) = P_0\,\delta\left(x-\frac{L}{2}\right)$

$$\int_0^L \sin\frac{i\pi x}{L} P_0\,\delta\left(x-\frac{L}{2}\right)dx = P_0 \sin\frac{i\pi\frac{L}{2}}{L} =$$

$$P_0 \sin i\frac{\pi}{2}$$

$$W(x,t) = \sum_{i=1}^{\infty} \frac{2}{mL} P_0 \sin i\frac{\pi}{2} \frac{\sin\frac{i\pi x}{L}\sin\omega t}{\omega_i^2-\omega^2}$$

$$\sum_{i=1}^{\infty} \frac{2P_0}{mL}(-1)^{i+1} \frac{\sin\frac{(2i-1)\pi x}{L}\sin\omega t}{\omega_i^2-\omega^2}$$

$$W\left(\frac{L}{2},t\right) = \sum_{i=1}^{\infty} \frac{2P_0}{mL} \frac{\sin\omega t}{\omega_i^2-\omega^2}$$

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MECHANICAL VIBRATIONS

Exercise 6: Continuous systems and Approximative methods

Problem solutions

1. We calculate the surface gradient:

$$\nabla_s(f\underline{A}) = \frac{\partial}{\partial\theta_1}(f\underline{A})\otimes\bar{t}^1 + \frac{\partial}{\partial\theta_2}(f\underline{A})\otimes\bar{t}^2 =$$

$$\left(\frac{\partial f}{\partial\theta_1}\underline{A} + f\frac{\partial \underline{A}}{\partial\theta_1}\right)\otimes\bar t^1 + \left(\frac{\partial f}{\partial\theta_2}\underline{A}+f\frac{\partial\underline A}{\partial\theta_2}\right)\otimes\bar t^2 =$$

$$f\left(\frac{\partial \underline A}{\partial\theta_1}\otimes\bar t^1 + \frac{\partial \underline A}{\partial\theta_2}\otimes \bar t^2\right) + \underline A\otimes\left(\frac{\partial f}{\partial\theta_1}\bar t^1+\frac{\partial f}{\partial\theta_2}\bar t^2\right) =$$

$$f\nabla_s\underline A + \underline A\otimes\nabla_s f$$

and then

$$div_s(f\underline A) = f\, div_s \underline A + \underline A \nabla_s f$$

From this we get

$$div_s(T_0\underline{1}_s) = T_0\, div_s \underline 1_s + \underline 1_s \nabla_s T_0 =$$

$$T_0\, div_s \underline 1_s + \nabla_s T_0$$

But $div_s \underline 1_s = div_s(\bar t_1\otimes\bar t^1 + \bar t_2\otimes\bar t^2) =$

$$div_s(\underline 1 - \bar n\otimes\bar n) = -\,div_s\, \bar n\otimes\bar n = -\left(\frac{\partial}{\partial\theta_1}(\bar n\otimes\bar n)\bar t^1 + \right.$$

$$\left.+\frac{\partial}{\partial\theta_2}(\bar n\otimes\bar n)\bar t^2\right) = -\left(\frac{\partial\bar n}{\partial\theta_1}\underbrace{\bar n\cdot\bar t^1}_{=0} + \bar n\frac{\partial\bar n}{\partial\theta_1}\cdot\bar t^1 + \right.$$

$$\left.\frac{\partial\bar n}{\partial\theta_2}\underbrace{\bar n\cdot\bar t^2}_{=0} + \bar n\frac{\partial \bar n}{\partial\theta_2}\cdot\bar t^2\right) = \bar n\left(-\frac{\partial\bar n}{\partial\theta_1}\cdot\bar t^1 - \frac{\partial\bar n}{\partial\theta_2}\cdot\bar t^2\right) =$$

$$\bar n(\bar t^1\cdot\underline W\bar t_1 + \bar t^2\cdot\underline W\bar t_2) = \bar n\, tr(\underline W) = \bar n\, 2\kappa_g$$

where we have used the Weingarten map

$$\underline W\bar t_1 = -\frac{\partial\bar n}{\partial\theta_1}, \quad \underline W\bar t_2 = -\frac{\partial\bar n}{\partial\theta_2}, \quad tr(\underline W) = 2\kappa_g$$

Thus

$$div_s(T_0\underline 1_s) = T_0\, 2\kappa_m\bar n + \nabla_s T_0$$

#

2. $\underline T_0 = \bar e_z\otimes\bar e_x S_x + \bar e_z\otimes\bar e_y S_y$

$$div_{s_0}\underline T_0 = \frac{\partial \underline T_0}{\partial x}\bar e_x + \frac{\partial\underline T_0}{\partial y}\bar e_y = \left(\bar e_z\otimes\bar e_x\frac{\partial S_x}{\partial x} + \right.$$

$$\left.\bar e_z\otimes\bar e_y\frac{\partial S_y}{\partial x}\right)\bar e_x + \left(\bar e_z\otimes\bar e_x\frac{\partial S_x}{\partial y} + \bar e_z\otimes\bar e_y\frac{\partial S_y}{\partial y}\right)\bar e_y =$$

$$\bar e_z\frac{\partial S_x}{\partial x} + \bar e_z\frac{\partial S_y}{\partial y} = \bar e_z\left(\frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}\right)$$

#

3. $\underline M_0 = -M_{xy}\bar e_x\otimes\bar e_x - M_y\bar e_x\otimes\bar e_y + M_x\bar e_y\otimes\bar e_x +$ $M_{xy}\bar e_y\otimes\bar e_y$

$$div_{s_0}\underline M_0 = \frac{\partial\underline M_0}{\partial x}\bar e_x + \frac{\partial \underline M_0}{\partial y}\bar e_y = \left(-\frac{\partial M_{xy}}{\partial x}\bar e_x\otimes\bar e_x - \right.$$

$$\frac{\partial M_y}{\partial x}\bar{e}_x\otimes\bar{e}_y + \frac{\partial M_x}{\partial x}\bar{e}_y\otimes\bar{e}_x + \frac{\partial M_{xy}}{\partial x}\bar{e}_y\otimes\bar{e}_y)\bar{e}_x +$$

$$(-\frac{\partial M_{xy}}{\partial y}\bar{e}_x\otimes\bar{e}_x - \frac{\partial M_y}{\partial y}\bar{e}_x\otimes\bar{e}_y + \frac{\partial M_x}{\partial y}\bar{e}_y\otimes\bar{e}_x +$$

$$\frac{\partial M_{xy}}{\partial y}\bar{e}_y\otimes\bar{e}_y)\bar{e}_y = -\frac{\partial M_{xy}}{\partial x}\bar{e}_x + \frac{\partial M_x}{\partial x}\bar{e}_y -$$

$$\frac{\partial M_y}{\partial y}\bar{e}_x + \frac{\partial M_{xy}}{\partial y}\bar{e}_y = \bar{e}_x(-\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y}) +$$

$$\bar{e}_y(\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_x}{\partial x})$$

#

4.a) The membrane equation:

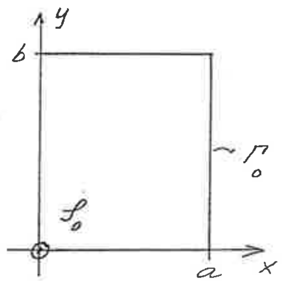
$$p + T_0 \Delta W = \ddot{W} m \qquad (1)$$

Displacement: $W = W(x,y,t)$

Free vibrations: $p \equiv 0$

b) Boundary conditions:

$$W(x,y,t) = 0 \qquad (x,y) \in \Gamma_0$$



c) standing wave solution:

$$w(x,y,t) = \hat{w}(x,y)\,\phi(t)$$

This inserted into (1) gives $(p=0)$

$$T_0 \Delta \hat{w} \phi = \hat{w} \ddot{\phi} m \quad \Rightarrow$$

$$\frac{\Delta \hat{w}}{\hat{w}} = \frac{m}{T_0}\frac{\ddot{\phi}}{\phi} = -\beta^2$$

where $\beta^2 =$ constant. From this we get

$$\Delta \hat{w} + \beta^2 \hat{w} = 0$$

$$\ddot{\phi} + \underbrace{\frac{T_0}{m}\beta^2}_{=\omega^2}\phi = 0 \quad \Rightarrow$$

$$\phi = \phi(t) = a \sin \omega t + b \cos \omega t$$

in addition to the eigenvalue problem:

$$\Delta \hat{w} + \beta^2 \hat{w} = 0 \qquad (2)$$

$$\hat{w}(x,y) = 0, \quad (x,y) \in \Gamma_0$$

We now look for solutions to (2) on the form:

$$\hat{w}(x,y) = X(x)\,Y(y)$$

This implies (insertion into (2),)

$$\frac{d^2X}{dx^2}Y + X\frac{d^2Y}{dy^2} + \beta^2 XY = 0$$

(divide through by XY) $\Rightarrow$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-\alpha^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-\gamma^2} + \beta^2 = 0$$

This leads to the equations

$$\frac{d^2X}{dx^2} + \alpha^2 X = 0 \qquad \frac{d^2Y}{dy^2} + \gamma^2 Y = 0 \qquad (3)$$

where

$$\alpha^2 + \gamma^2 = \beta^2 \qquad (4)$$

d) The solutions of (3) are:

$$X(x) = C_1 \sin\alpha x + C_2 \cos\alpha x$$

$$Y(y) = C_3 \sin\gamma y + C_4 \cos\gamma y$$

and then

$$\hat{w}(x,y) = X(x)\,Y(y) = A_1 \sin\alpha x \sin\gamma y + A_2 \sin\alpha x \cos\gamma y + A_3 \cos\alpha x \sin\gamma y + A_4 \cos\alpha x \cos\gamma y$$

where the constants $A_1, \dots, A_4$, α and γ are restricted by the boundary conditions.

$$\hat{w}(0,y) = A_3 \sin\gamma y + A_4 \cos\gamma y = 0 \quad \forall y \Rightarrow$$

$(\gamma \neq 0)$ $\quad A_3 = A_4 = 0$ (linear independence!)

$$\hat{w}(a,y) = A_1 \sin\alpha a \sin\gamma y + A_2 \sin\alpha a \cos\gamma y = 0$$

$$\Rightarrow \begin{cases} A_1 \sin\alpha a = 0 \\ A_2 \sin\alpha a = 0 \end{cases} \qquad (A_1, A_2) \neq (0,0) \Rightarrow$$

$$\sin\alpha a = 0 \Leftrightarrow \alpha a = m\pi, \quad m = 1, 2, \dots$$

$$\alpha_m = \frac{m\pi}{a} \qquad m = 1, 2, \dots$$

Similarly we get from

$$\hat{w}(x,0) = 0, \quad \hat{w}(x,b) = 0$$

$$A_2 = A_4 = 0 \quad \text{and} \quad \sin\gamma b = 0 \Leftrightarrow$$

$$\gamma_n = \frac{n\pi}{b} \qquad n = 1, 2, \dots$$

From the relation (4) it then follows

$$\beta = \beta_{mn} = \sqrt{\alpha_m^2 + \gamma_n^2} = \pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad m, n = 1, \dots$$

and then eigenfrequencies

$$\omega = \omega_{mn} = \sqrt{\frac{T_0}{m}}\,\beta_{mn} = \pi\sqrt{\frac{T_0}{m}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

Corresponding mode shapes are given by

$$\hat{w}_{mn}(x,y) = \frac{2}{\sqrt{mab}} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

mass density

5. <u>The thin plate equation:</u>

a) $D\Delta^2 w - p_0 + m\ddot{w} = 0$, $D = \dfrac{Eh^3}{12(1-\nu^2)}$ (1)

Displacement: $w = w(x,y,t)$

Free vibrations: $p_0 \equiv 0$

b) Boundary conditions: (simply supported)

$w(0,y) = 0$, $M_x(0,y) = 0 \iff \dfrac{\partial^2 w}{\partial x^2}(0,y) = 0$

$w(a,y) = 0$, $M_x(0,y) = 0 \iff \dfrac{\partial^2 w}{\partial x^2}(a,y) = 0$

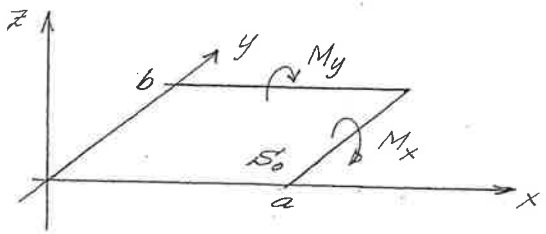
$w(x,0) = 0$, $M_y(x,0) = 0 \iff \dfrac{\partial^2 w}{\partial y^2}(x,0) = 0$

$w(x,b) = 0$, $M_y(x,b) = 0 \iff \dfrac{\partial^2 w}{\partial y^2}(x,b) = 0$

(t is omitted above). Note that:

$M_x = -D\left(\dfrac{\partial^2 w}{\partial x^2} + \nu\dfrac{\partial^2 w}{\partial y^2}\right)$, $w(0,y) = 0 \Rightarrow$

$\dfrac{\partial^2 w}{\partial y^2}(0,y) = 0 \Rightarrow M_x(0,y) = -D\dfrac{\partial^2 w}{\partial x^2}(0,y)$.



c) Free vibrations, standing wave solution:

$w = w(x,y,t) = \hat{w}(x,y)\sin\omega t$

Inserted into (1) $\Rightarrow$

$\Delta^2\hat{w} - \beta^4\hat{w} = 0$, $\beta^4 = \omega^2\dfrac{m}{D}$ (2)

Obviously we can factorize:

$\Delta^2 - \beta^4 = (\Delta+\beta^2)(\Delta-\beta^2) = (\Delta-\beta^2)(\Delta+\beta^2)$

Now let $\hat{w}_1$ and $\hat{w}_2$ satisfy

$(\Delta-\beta^2)\hat{w}_1 = 0$, $(\Delta+\beta^2)\hat{w}_2 = 0$

Then $\hat{w} = \hat{w}_1 + \hat{w}_2$ is a solution to (2)

$(\Delta^2-\beta^4)\hat{w} = (\Delta+\beta^2)(\Delta-\beta^2)(\hat{w}_1+\hat{w}_2) = 0$

Equation $(\Delta-\beta^2)\hat{w}_1 = 0$ has the solutions

$\hat{w}_1 = Ae^{\alpha x}e^{\gamma y}$, $\alpha^2+\gamma^2 = \beta^2$

and $(\Delta+\beta^2)\hat{w}_2 = 0$ the solution:

$\hat{w}_2 = Ae^{i\alpha x}e^{i\gamma y}$ $\alpha^2+\gamma^2 = \beta^2$

A general solution to (2) may then be written:

$$\hat{w}(x,y) = A_1 \sin\alpha x \sin\gamma y + A_2 \cos\alpha x \sin\gamma y +$$
$$A_3 \sin\alpha x \cos\gamma y + A_4 \cos\alpha x \cos\gamma y +$$
$$A_5 \sinh\alpha x \sinh\gamma y + A_6 \cosh\alpha x \sinh\gamma y +$$
$$A_7 \sinh\alpha x \cosh\gamma y + A_8 \cosh\alpha x \cosh\gamma y$$

where $A_1, A_2, \ldots, A_8, \alpha, \gamma$ are determined by the boundary conditions

d) Boundary conditions $\Rightarrow$

$$\hat{w}(0,y) = A_2 \sin\gamma y + A_4 \cos\gamma y + A_6 \sinh\gamma y +$$
$$A_8 \cosh\gamma y = 0 \quad \forall y \quad \Rightarrow \text{(linear independence)}$$

$$A_2 = A_4 = A_6 = A_8 = 0$$

$$\hat{w}(x,0) = A_3 \sin\alpha x + A_7 \sinh\alpha x = 0 \quad \forall x \quad \Rightarrow$$

$$A_3 = A_7 = 0$$

Thus

$$\hat{w}(x,y) = A_1 \sin\alpha x \sin\gamma y + A_5 \sinh\alpha x \sinh\gamma y$$

$$\frac{\partial^2 \hat{w}}{\partial x^2} = -A_1\alpha^2 \sin\alpha x \sin\gamma y + A_5\alpha^2 \sinh\alpha x \sinh\gamma y$$

$$\frac{\partial^2 \hat{w}}{\partial y^2} = -A_1\gamma^2 \sin\alpha x \sin\gamma y + A_5\gamma^2 \sinh\alpha x \sinh\gamma y$$

$$\frac{\partial^2 \hat{w}}{\partial x^2}(a,y) = -A_1\alpha^2 \sin\alpha a \sin\gamma y + A_5\alpha^2 \sinh\alpha a \sinh\gamma y$$

$$= 0 \quad \forall y \quad \Rightarrow \text{(linear independence)}$$

$$A_1\alpha^2 \sin\alpha a = 0 \,, \quad A_5\alpha^2 \sinh\alpha a = 0 \quad \Rightarrow$$

$$(\alpha \neq 0) \quad \sin\alpha a = 0 \,, \quad A_5 = 0$$

In the same way the condition $\frac{\partial^2 \hat{w}}{\partial y^2}(x,b) = 0$ implies

$$\sin\gamma b = 0 \,, \quad A_5 = 0$$

$$\sin\alpha a = 0 \quad \Rightarrow \quad \alpha = \alpha_m = \frac{m\pi}{a} \qquad m = 1, 2, \ldots$$

$$\sin\gamma b = 0 \quad \Rightarrow \quad \gamma = \gamma_n = \frac{n\pi}{b} \qquad n = 1, 2, \ldots$$

$$\beta^2 = \beta_{mn}^2 = \alpha_m^2 + \gamma_n^2 = \pi^2\left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right)$$

$$\beta^4 = \omega^2 \frac{m_0}{D} \quad \Rightarrow \quad \underline{\text{Eigenfrequencies:}}$$

$$\omega = \omega_{mn} \quad \pi^2 \sqrt{\frac{D}{m_0}} \left(\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right)$$

and corresponding <u>eigenmodes</u>

$$\hat{w}_{m,n}(x,y) = A \sin \frac{m\pi x}{a} \sin \frac{n\pi x}{b} \qquad m,n = 1,2,\dots$$

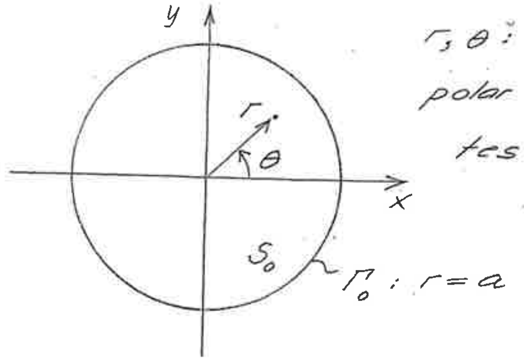
#

6.

a) $D\Delta^2 w - p_0 + m\ddot{w} = 0 \ , \quad D = \dfrac{Eh^3}{12(1-\nu^2)} \qquad (1)$

Displacement : $w = w(r,\theta,t)$

Free vibrations: $p_0 \equiv 0$



r, θ: plane polar coordinates.

b) "Clamped boundary":

$$w(a,\theta,t) = 0 \ , \quad \forall\theta$$

$$\frac{\partial w}{\partial r}(a,\theta,t) = 0 \qquad \left(\frac{\partial w}{\partial r} = \nabla w \cdot \bar{e}_r\right) , \quad \forall\theta$$

c) Standing wave solution:

$$w = w(r,\theta,t) = \hat{w}(r,\theta)\sin\omega t$$

Inserted into (1) $\Rightarrow$

$$\Delta^2\hat{w} - \beta^4\hat{w} = 0 \ , \quad \beta^4 = \omega^2\frac{m}{D}$$

$$\Delta^2 - \beta^4 = (\Delta^2 + \beta^2)(\Delta^2 + (i\beta)^2)$$

Look for solutions to

$$(\Delta^2 + \beta^2)\hat{w}_1 = 0 \ , \quad (\Delta^2 + (i\beta)^2)\hat{w}_2 = 0$$

In polar coordinates we have

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

Assuming a solution

$$\hat{w}_1(r,\theta) = R_1(r)\Theta_1(\theta)$$

$$(\Delta^2 + \beta^2)\hat{w}_1 = \left(\frac{d^2R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr}\right)\Theta_1 + \frac{R_1}{r^2}\frac{d^2\Theta_1}{d\theta^2} +$$

$$\beta^2 R_1\Theta_1 = 0$$

which can be separated into

$$\frac{d^2\Theta_1}{d\theta^2} + \varkappa^2 \Theta_1 = 0 \qquad (2)$$

$$\frac{d^2R_1}{dr^2} + \frac{1}{r}\frac{dR_1}{dr} + \left(\beta^2 - \frac{\varkappa^2}{r^2}\right)R_1 = 0 \qquad (3)$$

where $\varkappa^2$ is a constant. $(2) \Rightarrow$

$\Theta_1(\theta) = C_1 \sin \varkappa\theta + C_2 \cos \varkappa\theta$

Continuity $\Rightarrow$ $\Theta_1(\theta) = \Theta_1(\theta + n2\pi)$ $\forall \theta$

$n = 0, 1, \ldots$ $\Rightarrow$ $\varkappa$ must be an integer

$\varkappa = m \qquad m = 0, 1, \ldots$

and we have the solutions to (2):

$\Theta_{1m}(x) = C_1 \sin m\theta + C_2 \cos m\theta, \quad m = 0, 1, \ldots$

The solutions to (3) may now be written

$R_{1m}(r) = C_3 J_m(\beta r) + C_4 Y_m(\beta r), \quad m = 0, 1, \ldots$

where $J_m = J_m(\beta r)$ and $Y_m = Y_m(\beta r)$ are Bessel functions of the first and second kind. The general solution can be written in the form

$\hat{W}_{1m}(r,\theta) = A_1 J_m(\beta r) \sin m\theta + A_2 J_m(\beta r) \cos m\theta +$

$A_3 Y_m(\beta r) \sin m\theta + A_4 Y_m(\beta r) \cos m\theta$

(Note that these are the solutions for the free membrane vibrations as well!)

Now the equation: $(\Delta + (i\beta)^2)\hat{W}_2 = 0$ has a solution of the same structure where β is replaced by $i\beta$ and we obtain the "hyperbolic Bessel functions"

$I_m(\beta r) = J_m(i\beta r), \quad K_m(\beta r) = Y_m(i\beta r)$

and we then finally have the solution:

$$\begin{aligned}\hat{W}_m(r,\theta) = {} & (A_1 J_m(\beta r) + A_2 Y_m(\beta r) + A_3 I_m(\beta r) + \\ & A_4 K_m(\beta r)) \sin m\theta + (A_5 J_m(\beta r) + A_6 Y_m(\beta r) + \\ & A_7 I_m(\beta r) + A_8 K_m(\beta r)) \cos m\theta \qquad m = 0, 1, \ldots\end{aligned} \qquad (4)$$

Since Y_m and K_m become infinite at $r = 0$ we have to eliminate these functions from the solution.

Hence (4) is reduced to

$$\hat{W}_m(r,\theta) = (A_1 J_m(\beta r) + A_2 I_m(\beta r))\sin m\theta +$$

$$(A_3 J_m(\beta r) + A_4 I_m(\beta r))\cos m\theta$$

The boundary conditions in b) give:

$$A_1 J_m(\beta a) + A_2 I_m(\beta a) = 0$$

$$A_3 J_m(\beta a) + A_4 I_m(\beta a) = 0$$

and then:

$$\hat{W}_m(r,\theta) = \left(J_m(\beta r) - \frac{J_m(\beta a)}{I_m(\beta a)} I_m(\beta r)\right)(A_1 \sin m\theta +$$

$$A_2 \cos m\theta)$$

and ("the characteristic equation")

$$\frac{d}{dr}\left(J_m(\beta r) - \frac{J_m(\beta a)}{I_m(\beta a)} I_m(\beta r)\right)\Bigg|_{r=a} = 0 \qquad (5)$$

But

$$\frac{d}{dr} J_m(\beta r) = \beta\left[J_{m-1}(\beta r) - \frac{m}{\beta r} J_m(\beta r)\right]$$

$$\frac{d}{dr} I_m(\beta r) = \beta\left[I_{m-1}(\beta r) - \frac{m}{\beta r} I_m(\beta r)\right]$$

and then (5) reduces to

$$I_m(\beta a) J_{m-1}(\beta a) - J_m(\beta a) I_{m-1}(\beta a) = 0 \qquad (6)$$

For a given $m = 0, 1, \ldots$ (6) has the solutions

$$\beta = \beta_{mn}, \qquad n = 1, 2, \ldots$$

$$\beta^4 = \omega^2 \frac{m}{D} \Rightarrow \underline{\underline{\omega_{mn} = \beta_{mn}^2 \sqrt{\frac{D}{m}}}}$$

$m = 0 \Rightarrow$ <u>mode shapes:</u>

$$\hat{W} = \hat{W}_{0n}(r,\theta) = A(I_0(\beta_{0n} a) J_0(\beta_{0n} r) -$$

$$J_0(\beta_{0n} a) I_0(\beta_{0n} r))$$

$m > 0 \Rightarrow$ <u>mode shapes:</u>

$$\hat{W}_{mnc}(r,\theta) = A[I_m(\beta_{mn} a) J_m(\beta_{mn} r) -$$

$$J_m(\beta_{mn} a) I_m(\beta_{mn} r)]\cos m\theta$$

$$\hat{W}_{mns}(r,\theta) = A[I_m(\beta_{mn} a) J_m(\beta_{mn} r) -$$

$$J_m(\beta_{mn} a) I_m(\beta_{mn} r)]\sin m\theta$$

We thus have two modeshapes corresponding to the eigenvalue ω_{mn}, $m>0$

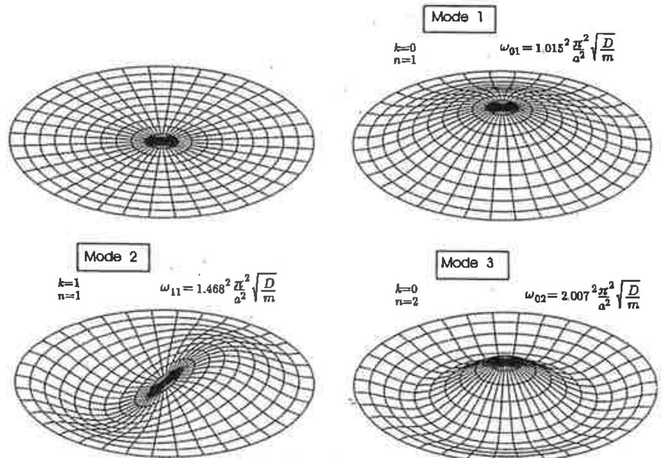


Figure 4.6.9
The first three eigenmodes of the clamped circular plate

#

7. With <u>basis functions</u>:

$$f_1(x) = \frac{x}{L} \quad , \quad f_2(x) = \left(\frac{x}{L}\right)^2$$

Let m denote the mass per unit length of the bar, then

$$m = \rho A$$

where A is the cross-sectional area of the bar.

We compute the <u>mass matrix</u>:

$$m_{11} = \int_0^L m\, f_1(x) f_1(x)\, dx = \int_0^L m \left(\frac{x}{L}\right)^2 d = \frac{mL}{3}$$

$$m_{12} = m_{21} = \int_0^L m \left(\frac{x}{L}\right)^3 dx = \frac{mL}{4}$$

$$m_{22} = \int_0^L m \left(\frac{x}{L}\right)^4 dx = \frac{mL}{5}$$

$$\underline{M} = mL \begin{pmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

and the <u>stiffness matrix</u>:

$$k_{11} = \int_0^L EA (f_1')^2 dx = \int_0^L EA \frac{1}{L^2}\, dx = \frac{EA}{L}$$

$$k_{12} = k_{21} = \int_0^L EA \frac{2x}{L^3}\, dx = \frac{EA}{L}$$

$$k_{22} = \int_0^L EA \frac{4x^2}{L^2}\, dx = \frac{4}{3} \frac{EA}{L}$$

$$\underline{K} = \frac{EA}{L} \begin{pmatrix} 1 & 1 \\ 1 & \frac{4}{3} \end{pmatrix}$$

The eigenvalue problem:

$$(-\omega^2 \underline{M} + \underline{K})\hat{\underline{q}} = \bar{0}\ , \quad \hat{\underline{q}} \neq \bar{0} \quad \Rightarrow$$

$$\det(-\omega^2 \underline{M} + \underline{K}) = 0 \quad \Leftrightarrow$$

$$3\mu^2 - 104\mu + 240 = 0 \qquad (1)$$

where

$$\mu = \omega^2 \frac{mL^2}{EA} = \omega^2 \frac{\rho A L^2}{EA} = \omega^2 L^2 \frac{\rho}{E}$$

The roots to (1):

$$\mu_1 = 2.49 \Rightarrow \underline{\omega_1} = \sqrt{\frac{2.49}{L^2}\frac{E}{\rho}} \cong \underline{1580\,s^{-1}}$$

$$\mu_2 = 32.18 \Rightarrow \underline{\omega_2} = \sqrt{\frac{32.18}{L^2}\frac{E}{\rho}} \cong \underline{5670\,s^{-1}}$$

Corresponding mode shapes:

$\underline{\omega_1 = 1580 s^{-1}}$: $\hat{q}_1 = \begin{pmatrix} 1 \\ -0.453 \end{pmatrix}$

$$u_1(x) = a\left(\frac{x}{L} - 0.453\left(\frac{x}{L}\right)^2\right), \quad a \neq 0$$

$\underline{\omega_2 = 5670 s^{-1}}$: $\hat{q}_2 = \begin{pmatrix} 1 \\ -1.381 \end{pmatrix}$

$$u_2(x) = a\left(\frac{x}{L} - 1.381\left(\frac{x}{L}\right)^2\right)$$

The exact eigenfrequencies and mode shapes

$$\omega_k = (2k-1)\frac{\pi}{2}\frac{1}{L}\sqrt{\frac{E}{\rho}} \qquad k = 1, 2, \ldots$$

$$u_k(x) = a \sin (2k-1)\frac{\pi x}{2L}$$

k=1:

$\omega_1 = 1570\,s^{-1}$

$u_1(x) = a \sin \frac{\pi x}{2L}$

k=2:

$\omega_2 = 4710\,s^{-1}$

$u_2(x) = a \sin \frac{3\pi x}{2L}$

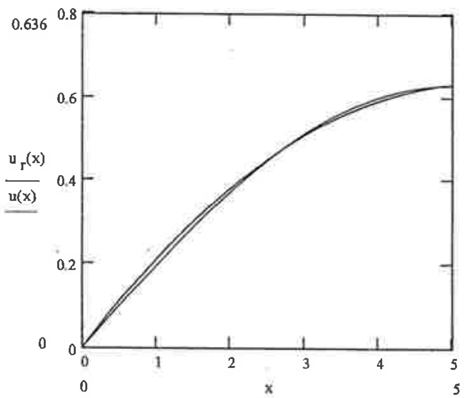
Comparison of first mode shapes:

Rayleigh-Ritz approximation

$$u_r(x) = 1.163\left(\frac{x}{L} - 0.453\left(\frac{x}{L}\right)^2\right)$$

Exact solution

$$u(x) = 0.632 \sin \frac{\pi x}{L}$$



Note that

$$\int_0^L u_r^2 dx =$$

$$\int_0^L u^2 dx = 1$$

8. With <u>basis functions</u>

$$f_1(x) = \left(\frac{x}{L}\right)^2 , \quad f_2(x) = \left(\frac{x}{L}\right)^3$$

We compute the <u>mass matrix</u>

$$m_{11} = \int_0^L m\, f_1(x)^2 dx = \int_0^L m \left(\frac{x}{L}\right)^4 dx = \frac{mL}{5}$$

$$m_{12} = m_{21} = \int_0^L m \left(\frac{x}{L}\right)^5 dx = \frac{mL}{6}$$

$$m_{22} = \int_0^L m \left(\frac{x}{L}\right)^6 dx = \frac{mL}{7}$$

$$\underline{M} = mL \begin{pmatrix} \frac{1}{5} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{7} \end{pmatrix}$$

and <u>stiffness matrix</u>:

$$k_{11} = \int_0^L EI\, f_1''(x)^2 dx = \frac{4EI}{L^3}$$

$$k_{12} = k_{21} = \int_0^L EI\, f_1''(x) f_2''(x)\, dx = \frac{6EI}{L^3}$$

$$k_{22} = \int_0^L EI\, f_2''(x)^2 dx = \frac{12EI}{L^3}$$

$$\underline{K} = \frac{EI}{L^3} \begin{pmatrix} 4 & 6 \\ 6 & 12 \end{pmatrix}$$

<u>The eigenvalue problem:</u>

$$(-\omega^2 \underline{M} + \underline{K})\hat{q} = \bar{0} , \quad \hat{q} \neq 0 \quad \Rightarrow$$

$$\det(-\omega^2 \underline{M} + \underline{K}) = 0 \quad \Leftrightarrow$$

$$\mu^2 - 1224\mu + 15120 = 0 \qquad (1)$$

$$\mu = \omega^2 \frac{mL^4}{EI}$$

The roots to (1):

$$\mu_1 = 12.48 \quad \Rightarrow \quad \underline{\omega_1^2 = 12.48 \frac{EI}{mL^4}}$$

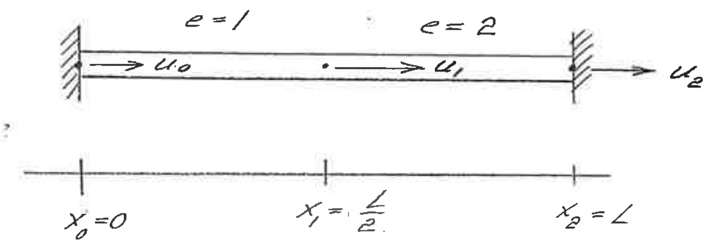
$$\mu_2 = 1212 \quad \Rightarrow \quad \underline{\omega_2^2 = 1212 \frac{EI}{mL^4}}$$

Exact eigenfrequencies:

$$\omega_1^2 = 12.36 \frac{EI}{mL^4}$$

$$\omega_2^2 = 494.6 \frac{EI}{mL^4}$$

9.



The structural mass- and stiffness-matrices for the bar (N = 2)

$$\underline{M} = \frac{m}{12}\begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

consistent mass

$$\underline{K} = \frac{2EA}{L}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

The clamped-clamped boundary conditions $u_0 = u_2 = 0$ reduces this to

$$M = \frac{m}{12} \cdot 4 = \frac{m}{3}$$

$$K = \frac{2EA}{L} \cdot 2 = \frac{4EA}{L}$$

and the eigenfrequency:

$$\omega^2 = \frac{K}{M} = \frac{12\,EA}{mL} \Rightarrow$$

$$\omega = \sqrt{12}\sqrt{\frac{EA}{mL}} \cong 3.464\sqrt{\frac{EA}{mL}}$$

With a lumped mass we get

$$\underline{M} = \frac{m}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and then employing boundary conditions

$$M = \frac{m}{4} \cdot 2 = \frac{m}{2}$$

and

$$\omega^2 = \frac{K}{M} = \frac{8EA}{mL} \Rightarrow \underline{\omega \cong 2.8284\sqrt{\frac{EA}{mL}}}$$

The exact first eigenfrequency is given by

$$\omega = \pi\sqrt{\frac{EA}{mL}}$$

Exercise 6.10

Input:

$m := 10$

$EI := 9.4 \cdot 10^{5}$

$L := 2.0$

CALCULATIONS:

$m_0 := \dfrac{m}{L}$ Mass density

Two element model:

$N := 2 \qquad l_e := \dfrac{L}{N}$

Localization operators:

$$\mathbb{L}_1 := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \mathbb{L}_2 := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{L}_1 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad \mathbb{L}_2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$$

Element mass matrix:

$$M_e := \frac{m_0 \cdot l_e}{420} \cdot \begin{pmatrix} 156 & 22 \cdot l_e & 54 & -13 \cdot l_e \\ 22 \cdot l_e & 4 \cdot l_e^{2} & 13 \cdot l_e & -3 \cdot l_e^{2} \\ 54 & 13 \cdot l_e & 156 & -22 \cdot l_e \\ -13 \cdot l_e & -3 \cdot l_e^{2} & -22 \cdot l_e & 4 \cdot l_e^{2} \end{pmatrix} \qquad M_e = \begin{pmatrix} 1.857 & 0.262 & 0.643 & -0.155 \\ 0.262 & 0.048 & 0.155 & -0.036 \\ 0.643 & 0.155 & 1.857 & -0.262 \\ -0.155 & -0.036 & -0.262 & 0.048 \end{pmatrix}$$

Element stiffness matrix:

$$K_e := \frac{EI}{l_e^{3}} \cdot \begin{pmatrix} 12 & 6 \cdot l_e & -12 & 6 \cdot l_e \\ 6 \cdot l_e & 4 \cdot l_e^{2} & -6 \cdot l_e & 2 \cdot l_e^{2} \\ -12 & -6 \cdot l_e & 12 & -6 \cdot l_e \\ 6 \cdot l_e & 2 \cdot l_e^{2} & -6 \cdot l_e & 4 \cdot l_e^{2} \end{pmatrix} \quad K_e = \begin{pmatrix} 1.128 \times 10^{7} & 5.64 \times 10^{6} & -1.128 \times 10^{7} & 5.64 \times 10^{6} \\ 5.64 \times 10^{6} & 3.76 \times 10^{6} & -5.64 \times 10^{6} & 1.88 \times 10^{6} \\ -1.128 \times 10^{7} & -5.64 \times 10^{6} & 1.128 \times 10^{7} & -5.64 \times 10^{6} \\ 5.64 \times 10^{6} & 1.88 \times 10^{6} & -5.64 \times 10^{6} & 3.76 \times 10^{6} \end{pmatrix}$$

Structure mass matrix assembly:

$$M := \sum_{i=1}^{N} \mathbb{L}_i^{T} \cdot M_e \cdot \mathbb{L}_i \qquad M = \begin{pmatrix} 1.857 & 0.262 & 0.643 & -0.155 & 0 & 0 \\ 0.262 & 0.048 & 0.155 & -0.036 & 0 & 0 \\ 0.643 & 0.155 & 3.714 & 0 & 0.643 & -0.155 \\ -0.155 & -0.036 & 0 & 0.095 & 0.155 & -0.036 \\ 0 & 0 & 0.643 & 0.155 & 1.857 & -0.262 \\ 0 & 0 & -0.155 & -0.036 & -0.262 & 0.048 \end{pmatrix}$$

Structural mass matrix, (Invoking boundary conditions, I.e. deleting appropriate rows and columns):

$$M := \begin{pmatrix} 3.714 & 0 \\ 0 & 0.095 \end{pmatrix}$$

Structure stifness matrix assembly: $K := \sum_{i=1}^{N} \mathbb{L}_i^{T} \cdot K_e \cdot \mathbb{L}_i$

$$K = \begin{pmatrix} 1.128 \times 10^{7} & 5.64 \times 10^{6} & -1.128 \times 10^{7} & 5.64 \times 10^{6} & 0 & 0 \\ 5.64 \times 10^{6} & 3.76 \times 10^{6} & -5.64 \times 10^{6} & 1.88 \times 10^{6} & 0 & 0 \\ -1.128 \times 10^{7} & -5.64 \times 10^{6} & 2.256 \times 10^{7} & 0 & -1.128 \times 10^{7} & 5.64 \times 10^{6} \\ 5.64 \times 10^{6} & 1.88 \times 10^{6} & 0 & 7.52 \times 10^{6} & -5.64 \times 10^{6} & 1.88 \times 10^{6} \\ 0 & 0 & -1.128 \times 10^{7} & -5.64 \times 10^{6} & 1.128 \times 10^{7} & -5.64 \times 10^{6} \\ 0 & 0 & 5.64 \times 10^{6} & 1.88 \times 10^{6} & -5.64 \times 10^{6} & 3.76 \times 10^{6} \end{pmatrix}$$

Structural mass matrix, (invoking boundary conditions, I.e deleting appropriate rows and columns):

$$K := \begin{pmatrix} 2.256\times10^{7} & 0 \\ 0 & 7.52\times10^{6} \end{pmatrix}$$

Finite element eigenvalues:

$$\Omega 2 := \mathrm{genvals}(K, M) \qquad \Omega 2 = \begin{pmatrix} 6.074\times10^{6} \\ 7.916\times10^{7} \end{pmatrix} \qquad \sqrt{\Omega 2} = \begin{pmatrix} 2.46461\times10^{3} \\ 8.89707\times10^{3} \end{pmatrix}$$

"Exact" eigenvalues:

$$\sqrt{500.55\cdot\frac{EI}{m_0\cdot L^4}} = 2.425\times10^{3} \qquad \sqrt{3803.1\cdot\frac{EI}{m_0\cdot L^4}} = 6.685\times10^{3}$$

Three element model:

$$N := 3 \qquad l_e := \frac{L}{N}$$

Localization operators:

$$\mathrm{IL}_1 := \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{pmatrix} \qquad \mathrm{IL}_2 := \begin{pmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix}$$

$$\mathrm{IL}_3 := \begin{pmatrix} 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \end{pmatrix}$$

$$\mathrm{IL}_1\cdot\begin{pmatrix}1\\2\\3\\4\\5\\6\\7\\8\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \qquad \mathrm{IL}_2\cdot\begin{pmatrix}1\\2\\3\\4\\5\\6\\7\\8\end{pmatrix} = \begin{pmatrix}3\\4\\5\\6\end{pmatrix} \qquad \mathrm{IL}_3\cdot\begin{pmatrix}1\\2\\3\\4\\5\\6\\7\\8\end{pmatrix} = \begin{pmatrix}5\\6\\7\\8\end{pmatrix}$$

Element mass matrix:

$$M_e := \frac{m_0\cdot l_e}{420}\cdot\begin{pmatrix} 156 & 22\cdot l_e & 54 & -13\cdot l_e \\ 22\cdot l_e & 4\cdot l_e^{2} & 13\cdot l_e & -3\cdot l_e^{2} \\ 54 & 13\cdot l_e & 156 & -22\cdot l_e \\ -13\cdot l_e & -3\cdot l_e^{2} & -22\cdot l_e & 4\cdot l_e^{2} \end{pmatrix} \qquad M_e = \begin{pmatrix} 1.238 & 0.116 & 0.429 & -0.069 \\ 0.116 & 0.014 & 0.069 & -0.011 \\ 0.429 & 0.069 & 1.238 & -0.116 \\ -0.069 & -0.011 & -0.116 & 0.014 \end{pmatrix}$$

Element stiffness matrix:

$$K_e := \frac{EI}{l_e^{3}}\cdot\begin{pmatrix} 12 & 6\cdot l_e & -12 & 6\cdot l_e \\ 6\cdot l_e & 4\cdot l_e^{2} & -6\cdot l_e & 2\cdot l_e^{2} \\ -12 & -6\cdot l_e & 12 & -6\cdot l_e \\ 6\cdot l_e & 2\cdot l_e^{2} & -6\cdot l_e & 4\cdot l_e^{2} \end{pmatrix} \quad K_e = \begin{pmatrix} 3.807\times10^{7} & 1.269\times10^{7} & -3.807\times10^{7} & 1.269\times10^{7} \\ 1.269\times10^{7} & 5.64\times10^{6} & -1.269\times10^{7} & 2.82\times10^{6} \\ -3.807\times10^{7} & -1.269\times10^{7} & 3.807\times10^{7} & -1.269\times10^{7} \\ 1.269\times10^{7} & 2.82\times10^{6} & -1.269\times10^{7} & 5.64\times10^{6} \end{pmatrix}$$

Structure mass matrix assembly: $M := \sum_{i=1}^{N} \mathrm{IL}_i^{T}\cdot M_e\cdot \mathrm{IL}_i$

$$M = \begin{pmatrix} 1.238 & 0.116 & 0.429 & -0.069 & 0 & 0 & 0 & 0 \\ 0.116 & 0.014 & 0.069 & -0.011 & 0 & 0 & 0 & 0 \\ 0.429 & 0.069 & 2.476 & 0 & 0.429 & -0.069 & 0 & 0 \\ -0.069 & -0.011 & 0 & 0.028 & 0.069 & -0.011 & 0 & 0 \\ 0 & 0 & 0.429 & 0.069 & 2.476 & 0 & 0.429 & -0.069 \\ 0 & 0 & -0.069 & -0.011 & 0 & 0.028 & 0.069 & -0.011 \\ 0 & 0 & 0 & 0 & 0.429 & 0.069 & 1.238 & -0.116 \\ 0 & 0 & 0 & 0 & -0.069 & -0.011 & -0.116 & 0.014 \end{pmatrix}$$

Structural mass matrix, (invoking boundary conditions, i.e. deleting appropriate rows and columns):

$$M := \begin{pmatrix} 2.476 & 0 & 0.429 & -0.069 \\ 0 & 0.028 & 0.069 & -0.011 \\ 0.429 & 0.069 & 2.476 & 0 \\ -0.069 & -0.011 & 0 & 0.028 \end{pmatrix}$$

Structure mass matrix assembly: $K := \sum_{i=1}^{N} \mathrm{IL}_i^{T}\cdot K_e\cdot \mathrm{IL}_i$

$$K = \begin{pmatrix} 3.807\times10^{7} & 1.269\times10^{7} & -3.807\times10^{7} & 1.269\times10^{7} & 0 & 0 & 0 & 0 \\ 1.269\times10^{7} & 5.64\times10^{6} & -1.269\times10^{7} & 2.82\times10^{6} & 0 & 0 & 0 & 0 \\ -3.807\times10^{7} & -1.269\times10^{7} & 7.614\times10^{7} & 0 & -3.807\times10^{7} & 1.269\times10^{7} & 0 & 0 \\ 1.269\times10^{7} & 2.82\times10^{6} & 0 & 1.128\times10^{7} & -1.269\times10^{7} & 2.82\times10^{6} & 0 & 0 \\ 0 & 0 & -3.807\times10^{7} & -1.269\times10^{7} & 7.614\times10^{7} & 0 & -3.807\times10^{7} & 1.269\times10^{7} \\ 0 & 0 & 1.269\times10^{7} & 2.82\times10^{6} & 0 & 1.128\times10^{7} & -1.269\times10^{7} & 2.82\times10^{6} \\ 0 & 0 & 0 & 0 & -3.807\times10^{7} & -1.269\times10^{7} & 3.807\times10^{7} & -1.269\times10^{7} \\ 0 & 0 & 0 & 0 & 1.269\times10^{7} & 2.82\times10^{6} & -1.269\times10^{7} & 5.64\times10^{6} \end{pmatrix}$$

Structural mass matrix, (Invoking boundary conditions, i.e deleting appropriate rows and columns):

$$K := \begin{pmatrix} 7.614\times10^{7} & 0 & -3.807\times10^{7} & 1.269\times10^{7} \\ 0 & 1.128\times10^{7} & -1.269\times10^{7} & 2.82\times10^{6} \\ -3.807\times10^{7} & -1.269\times10^{7} & 7.614\times10^{7} & 0 \\ 1.269\times10^{7} & 2.82\times10^{6} & 0 & 1.128\times10^{7} \end{pmatrix}$$

Finite element eigenvalues:

$$\Omega2 := \mathrm{genvals}(K, M) \qquad \Omega2 = \begin{pmatrix} 1.037\times10^{9} \\ 2.503\times10^{8} \\ 4.651\times10^{7} \\ 5.927\times10^{6} \end{pmatrix} \qquad \sqrt{\Omega2} = \begin{pmatrix} 3.22071\times10^{4} \\ 1.58217\times10^{4} \\ 6.82019\times10^{3} \\ 2.43459\times10^{3} \end{pmatrix}$$

"Exact" eigenvalues:

$$\sqrt{500.55\cdot\frac{EI}{m_0\cdot L^4}} = 2.425\times10^{3} \qquad \sqrt{3803.1\cdot\frac{EI}{m_0\cdot L^4}} = 6.685\times10^{3}$$

$$\sqrt{14620\cdot\frac{EI}{m_0\cdot L^4}} = 1.311\times10^{4} \qquad \sqrt{39944\cdot\frac{EI}{m_0\cdot L^4}} = 2.166\times10^{4}$$

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