LUND
UNIVERSITY
Mechanics, LTH

Mechanical Vibrations, FMEN10

2011-12-13, 14.00-19.00

Name (print):..

Personal id-number:...................................

Written examination with five examination tasks. Please check that all tasks are included. A clean copy of the solutions should be handed in. Write your name on all pages, number the pages and hand them in, in the correct order. Make sure that the papers are stapled together.

Summaries, equations-sheets and mathematical handbooks such as TEFYMA may be used.

Results:

Task	Comment	Points (0-3)
1		
2		
3		
4		
5		
	Sum	
	Grade	

Name (signature):... Leg:......

1)

a) The vibrational motion of a bar in extension is governed by the differential equation:

$$\frac{\partial}{\partial x}(EA(x)\frac{\partial u(x,t)}{\partial x})+k(x)=m(x)\frac{\partial^2 u(x,t)}{\partial t^2},\quad 0<x<L,\ t\geq 0 \qquad (1)$$

where L is the length of the bar, E is the modulus of elasticity, $A=A(x)$ is the sectional area, $k=k(x)$ is the external force density, $m=m(x)$ the mass density, and by the boundary-initial conditions:

$$\begin{aligned}&\alpha_1 u(0,t)+\beta_1\frac{\partial u(0,t)}{\partial x}=0,\quad \alpha_2 u(L,t)+\beta_2\frac{\partial u(L,t)}{\partial x}=0\\ &u(x,0)=u_0(x),\quad \frac{\partial u(x,0)}{\partial t}=\dot{u}_0(x)\end{aligned} \qquad (2)$$

where $\alpha_1,\beta_1,\alpha_2,\beta_2$ are given constants ($(\alpha_1,\beta_1)\neq(0,0)$ and $(\alpha_2,\beta_2)\neq(0,0)$) and $u_0=u_0(x)$ and $\dot{u}_0=\dot{u}_0(x)$ are given functions.

Assume that $k\equiv 0$ ("free vibrations") and consider a solution on the form ("separation of variables")

$$u(x,t)=\hat{u}(x)\phi(t) \qquad (3)$$

Show that the following conditions (4) and (5) are valid:

$$-\frac{1}{m(x)}\frac{d}{dx}(EA(x)\frac{d\hat{u}(x)}{dx})=\lambda\hat{u}(x) \qquad (4)$$

$$\frac{d^2}{dt^2}\phi(t)+\lambda\phi(t)=0 \qquad (5)$$

where λ is a constant.

b) A vibrating mechanical system may be described by the following mass matrix $\underline{M}$, damping matrix $\underline{C}$, and stiffness matrix $\underline{K}$.

$$\underline{M}=\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix},\ \underline{C}=\begin{pmatrix}50&0&0\\0&0&0\\0&0&0\end{pmatrix},\ \underline{K}=\begin{pmatrix}3000&-1000&-1000\\-1000&3000&-1000\\-1000&-1000&3000\end{pmatrix}$$

Is this problem diagonalizable? Motivate your answer.

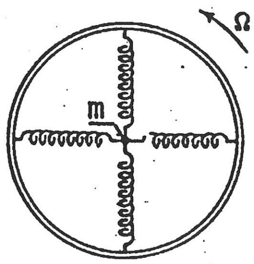
c) Consider a mechanical system $\mathcal{B}$ whose configuration space is 1-dimensional, i.e. the position vector of a material point $P \in \mathcal{B}$ is given by $\mathbf{r} = \mathbf{r}(q;P)$ where q is a *one* scalar variable, $q = q(t)$. We assume that the system has a potential energy $U = U(q)$ and that $Q^{ext} = Q^{int} = 0$.

Show that the kinetic energy of the system may be written

$$T(q,\dot{q}) = \frac{1}{2}a(q)\dot{q}^2 \text{ where } a(q) = \int_{\mathcal{B}} \frac{\partial \mathbf{r}}{\partial q} \cdot \frac{\partial \mathbf{r}}{\partial q} dm$$

d) For the same system as described in 1c): With the Lagrangian $L = L(q,\dot{q}) = T - U$ calculate the equation of motion for the system.

e) A particle with mass m is connected to a circular ring with four linear elastic springs. The ring is rotating in a plane, around its symmetry axis perpendicular to the plane and with a constant angular velocity Ω. When the particle is positioned at the centre of the ring the springs form a perpendicular cross according to the figure below. The generalized coordinates; x and y denote the displacement of the particle in two perpendicular directions. It is presumed that the particle may only move in the plane of the circle. Formulate the expression for the particle velocity using the coordinates x and y introduced in a coordinate system rotating with the ring.



Use the kinematical relations $\dot{\boldsymbol{e}}_x = \boldsymbol{\Omega} \times \boldsymbol{e}_x$, $\dot{\boldsymbol{e}}_y = \boldsymbol{\Omega} \times \boldsymbol{e}_y$ where $\boldsymbol{\Omega} = \boldsymbol{e}_z \Omega$.

f) A horizontal, simply supported beam (both ends), with density = ρ, modulus of elasticity = E, length = L, sectional area = A and sectional area moment-of-inertia = I (assumed to be constant), is subjected to an external transverse force

$$p = p(x,t) = \hat{p}(x)\sin\omega t, \quad \hat{p}(x) = p_0 = const.$$

The forced "steady-state" response of the midpoint $x = \frac{L}{2}$ of the beam is to be calculated. The mode shape has been found to be equal to

$$\hat{w}_i = \sin(\frac{i\pi x}{L})$$

Show how to use the orthonormality condition to rewrite

$$\sum_{i=1}^{\infty}(\omega_i^2\eta_i + \ddot{\eta}_i)\hat{w}_i = \frac{\hat{p}(x)}{m}\sin\omega t$$

so that the sum is eliminated from the expression above.

2)

a) An n-dimensional mechanical system with positive definite mass matrix $\underline{M}$ and positive semi-definite stiffness matrix $\underline{K}$ has the eigenfrequencies $\omega_1,\ \omega_2,\ \ldots,\omega_n$ and the modal matrix $\underline{X} = [\bar{x}_1\ \ldots\ \bar{x}_n]$ where

$$(-\omega_i^2\underline{M} + \underline{K})\bar{x}_i = \bar{0}, \quad i = 1,\ldots,n \tag{1}$$

and

$$\underline{X}^T\underline{M}\,\underline{X} = \underline{\mu} = diag(\mu_1\ \ldots\ \mu_n)\ and\ \underline{X}^T\underline{K}\,\underline{X} = \underline{\kappa} = diag(\kappa_1\ \ldots\ \kappa_n) \tag{2}$$

where $\mu_1,\ \ldots,\mu_n$ and $\kappa_1,\ \ldots,\kappa_n$ are constants satisfying $\mu_i > 0$ and $\kappa_i \geq 0$. The kinetic T and potential energies V of the system are given by

$$T = \frac{1}{2}\dot{\bar{q}}^T\underline{M}\dot{\bar{q}} \text{ and } V = \frac{1}{2}\bar{q}^T\underline{K}\bar{q} \tag{3}$$

Using *normal coordinates* $\bar{\eta} = (\eta_1\quad \eta_2\quad \cdot\ \cdot\ \cdot\quad \eta_n)^T$, defined by

$$\bar{q} = \underline{X}\bar{\eta} \tag{4}$$

Using the equations given above, show that the potential energy may be written

$$V = \frac{1}{2}\bar{\eta}^T\underline{\kappa}\bar{\eta} = \frac{1}{2}\sum_{i=1}^{n}\kappa_i\eta_i^2 \tag{5}$$

Include a comment on why it is possible to write V as a sum.

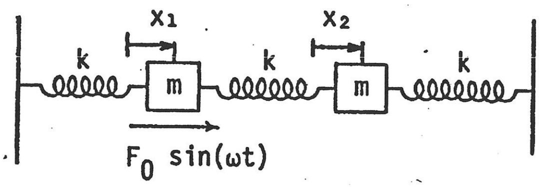
b) Consider the mechanical system in the figure below. The masses move along a fixed horizontal line and the left mass is subjected to an external harmonic force with amplitude F_0 and angular frequency ω. The mechanical system has the following mass and stiffness matrix

$$\underline{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \underline{K} = \begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}$$

and mode shapes

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; \bar{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Calculate the modal mass matrix.



c) A mechanical system consists of ten masses $m_1, m_2, \ldots, m_{10}$ where each mass m_n is connected to the next mass $m_{(n+1)}$ using a spring with spring constant k_n, $n=1,2,\ldots,9$. Observe that mass m_1 is only connected to mass m_2 and that mass m_{10} is only connected to mass m_9. Calculations show that the system has the following natural eigenfrequencies:

$$\omega_{01} = 41.1,\ \omega_{02} = 79.4,\ \omega_{03} = 115.8,\ \omega_{04} = 157.3,\ \omega_{05} = 279.4,$$
$$\omega_{06} = 333.6,\ \omega_{07} = 577.9,\ \omega_{08} = 645.2,\ \omega_{09} = 792.5,\ \omega_{010} = 839.4$$

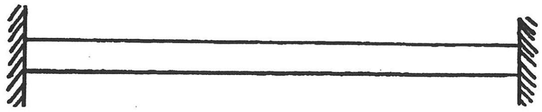
Referring to the numerical values of the eigenfrequencies, why is it reasonable to assume that the calculations are incorrect?

d) A vibrating mechanical system described by the coordinates x_1, x_2 and x_3 has the following linearized equations of motion

$$(k_1 + k_4 + k_6)x_1 - k_4 x_2 - k_6 x_3 + c_1 \dot{x}_1 + m_1 \ddot{x}_1 = 0$$
$$-k_5 x_3 - k_4 x_1 + m_2 \ddot{x}_2 + (k_2 + k_4 + k_5)x_2 = 0$$
$$m_3 \ddot{x}_3 - k_6 x_1 + (k_3 + k_5 + k_6)x_3 - k_5 x_2 = 0$$

with constants m_1, m_2, m_3, k_1, k_2, k_3, k_4, k_5, k_6 and c_1. Put the equations on matrix format and identify the mass matrix, the stiffness matrix, the damping matrix and the gyroscopic matrix.

e) A homogeneous bar is clamped at both its ends. The bar is subject to longitudinal vibrations, and the Finite Element method using two elements of equal size is chosen for calculation of the first natural frequency.



The following constant properties are known: the length of the bar: L, the sectional area: A, the sectional area moment-of-inertia: I, the material density: ρ and the modulus of elasticity: E. Using the element mass matrix, insert boundary conditions and assemble the structural mass matrix.

f) A mechanical system is modelled using the generalized coordinates r and φ. In the model m, ω, a, k and r_n are constants. The equations of motion of the system may be written

$$m\ddot{r} - mr\dot{\varphi}^2 - m\omega^2 r - 2m\omega r\dot{\varphi} - m\omega^2 a\cos\varphi + kr - kr_n = 0$$
$$2mr\dot{r}\dot{\varphi} + mr^2\ddot{\varphi} + 2m\omega r\dot{r} + m\omega^2 ar\sin\varphi = 0$$

Formulate stability conditions for the equilibrium state $r = r_0$, $\varphi = \varphi_0$. The system fulfils the conditions required for using the modified potential V^* when studying stability.

3) A mechanical system is defined by its mass matrix $\mathbf{M}$, its stiffness matrix $\mathbf{K}$ and its damping matrix $\mathbf{C}$ where

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}, \quad \alpha,\beta \text{ (positive) real constants}$$

The free vibrations are given by the solution to the initial value problem:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$
$$\mathbf{q}(0) = \mathbf{q}_0, \ \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0$$

where $\mathbf{q}_0, \dot{\mathbf{q}}_0$ are given constant vectors.

Show that the solution to this problem is given by:

$$\mathbf{q}(t) = \left[\sum_{i=1}^{n} e^{-\varsigma_i \omega_{0i} t}\left(\frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i}\cos(\omega_{di} t) + \varsigma_i \frac{\omega_{0i}}{\omega_{di}} \frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i}\sin(\omega_{di} t)\right)\right]\mathbf{q}_0 +$$
$$+\left[\sum_{i=1}^{n} e^{-\varsigma_i \omega_{0i} t} \frac{\mathbf{x}_i \mathbf{x}_i^T \mathbf{M}}{\mu_i \omega_{di}}\sin(\omega_{di} t)\right]\dot{\mathbf{q}}_0$$

where $\mathbf{x}_i$ denotes the classical mode shapes, $\mu_i, \omega_{0i}, \varsigma_i$ and ω_{di} are the modal mass, un-damped natural frequency, relative damping and damped natural frequency respectively, corresponding to the i^{th} mode. We assume that $0 \leq \varsigma_i < 1$ (weak damping).

4) A mechanical system consists of a particle pendulum with mass m_1 and cord-length l connected to a carriage, with mass m_2, which may move along a vertical guide, see figure below. The carriage is supported by a non-linear spring element with a retracting spring force equal to $S = S(x) = k_1 x + k_2 x^3,\ \ k_1, k_2 > 0$ where x is the elongation of the spring (measured from the unstressed state).

a) Using the generalized coordinates x and θ formulate the equations of motion for the system.

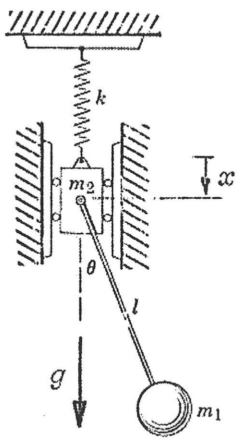
b) Describe how to calculate the un-damped natural frequencies and the corresponding mode shapes.

c) It turns out that the model is insufficient to describe the physical system which is to be investigated. In the model, a viscous damper with damping constant c is placed next to the spring. Formulate the equations of motion with this new addition of damping.

d) Describe how to calculate the modal relative dampings, the damped natural frequencies and the corresponding mode shapes.

e) Describe how to calculate the component $F_{22} = F_{22}(\omega)$ of the frequency response matrix.

In your description, please include the equations that have to be solved and present a list of names of all variables that are introduced in your equations. Necessary relationships that have to be used to solve the problem should be presented. Matlab and similar programmes are just tools. Your task is to present a systematic description for the reader how to solve b), d) and e) using the results found in a) and c).



5) In order to measure forces exerted by living nerve cells on their environment, nano wires are used where the deflections of the wires are measured. In order to calculate the force, physical properties of the nano wires need to be recorded. A first step is to formulate a mathematical model of the vibration of the nano wires. Formulate an expression for the mode shapes of transverse (bending) vibrations of a nano wire which is clamped (fixed) at one end and free at the other end. Also formulate an expression from which it is possible to numerically calculate the natural frequencies for the nano wires. The expressions may include mathematical functions such as for example sine and cosine, as well as the length of the nano wire: L, the sectional area: A, the sectional area moment-of-inertia: I, the material density: ρ and the modulus of elasticity: E. With given numerical values for the constants it should be possible to plot the theoretical mode shapes using your expression.