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MECHANICAL VIBRATIONS - Examination task 1

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Problem 1:1

Let $q = q_1, \dots, q_n$ be generalized coordinates for a material system B subjected to the generalized forces $Q_i = \int_B \mathbf{f}_a \cdot \frac{\partial \mathbf{r}}{\partial q_i} dm$, where \mathbf{f}_a is the (specific) active accelerating force. The position vector of a material point $P \in B$ is then given by $\mathbf{r} = \mathbf{r}(q; P)$. Let $\mathbf{a} = \mathbf{a}(q, \dot{q}, \ddot{q}; P)$ denote the acceleration of the material point.

a) Show that

$$\mathbf{a} = \sum_{j=1}^n \frac{\partial \mathbf{r}}{\partial q_j} \ddot{q}_j + \sum_{j=1}^n \sum_{k=1}^n \frac{\partial^2 \mathbf{r}}{\partial q_j \partial q_k} \dot{q}_j \dot{q}_k \tag{1}$$

b) Show that the equations of motion may be written

$$\sum_{j=1}^n a_{ij} \ddot{q}_j + \sum_{j=1}^n \sum_{k=1}^n C_{ijk} \dot{q}_j \dot{q}_k = Q_i \quad i = 1, \dots, n \tag{2}$$

where a_{ij} and C_{ijk} are defined by

$$a_{ij} = \int_B \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial \mathbf{r}}{\partial q_j} dm, \quad C_{ijk} = \int_B \frac{\partial \mathbf{r}}{\partial q_i} \cdot \frac{\partial^2 \mathbf{r}}{\partial q_j \partial q_k} dm \tag{3}$$

(Hint: Use d'Alembert's principle, equation (4.5) on page 31 in the Lecture Notes).

c) Show that if $\mathbf{f}_a = \mathbf{g} - c\mathbf{v}\mathbf{v}$ where \mathbf{g} is a constant vector, $\mathbf{v} = \dot{\mathbf{r}}$, $v = |\mathbf{v}|$ and $c > 0$ is a constant, then

$$Q_i = -\frac{\partial V}{\partial q_i} - \frac{\partial D}{\partial \dot{q}_i} \tag{4}$$

where

$$V = -m\mathbf{g} \cdot \mathbf{r}_c \quad \text{and} \quad D = \int_B \frac{cv^3}{3} dm \tag{5}$$

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and \mathbf{r}_c is the position vector of the *centre of mass* c of the body and m is its total mass. (Hint: Use relation (4.7) on page 32 in the Lecture Notes).

Solution:

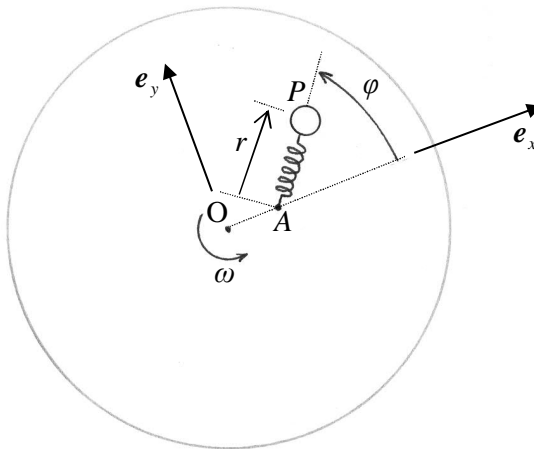
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Problem 1:2

A particle P , which may slide *without friction* on a *horizontal* table, is connected to a fixed point A on the table with a linear elastic spring, see figure below. The table is rotating around a vertical axis through O with the *constant* angular velocity ω . Distance $OA = a$. Particle mass = m . Spring constant = k and spring natural (unstressed) length = r_n . Using coordinates r and φ

- Give an expression for the position vector of the particle P : $\mathbf{r} = \mathbf{r}(r, \varphi; P)$. (Hint: use the orthonormal base $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ fixed to the rotating table, see figure below).
- Derive an expression for the velocity $\mathbf{v} = \dot{\mathbf{r}}$ of the particle. (Hint: use the kinematical relations $\dot{\mathbf{e}}_x = \boldsymbol{\omega} \times \mathbf{e}_x$, $\dot{\mathbf{e}}_y = \boldsymbol{\omega} \times \mathbf{e}_y$ where $\boldsymbol{\omega} = \omega \mathbf{e}_z$).
- Derive the kinetic energy $T = T(r, \varphi, \dot{r}, \dot{\varphi})$ for the system. Give the expressions for T_0 , T_1 and T_2 .
- Derive the potential energy $V = V(r, \varphi)$ for the system.
- Derive (by using Lagrange's method) the equations of motion for the system.
- Determine the (relative) *equilibrium states* ($r(t) = r_o$, $\varphi(t) = \varphi_o$, $r_o > 0$ and φ_o are constants) of the particle. (Note that the existence of equilibrium states will depend on the relation between the parameters: ω , a , m , k and r_n).
- Consider the modified potential energy $V^* = V - T_0$. Characterize the modified potential energy at the equilibrium states (maximum, minimum or indifferent?).
- Linearize the equations of motion at the equilibrium state where the modified potential energy has a minimum. Put the equation on matrix format and identify the mass matrix, the stiffness matrix, the damping matrix and the gyroscopic matrix. (Hint: For the linearization use the Maclaurin series for trigonometric functions, i.e.

$$\sin x = x - \frac{x^3}{3!} + \dots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2!} + \dots$$



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Solution:

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Problem 2:1

An n -dimensional mechanical system with positive definite mass matrix \underline{M} and positive semi-definite stiffness matrix \underline{K} has the eigenfrequencies $\omega_1, \omega_2, \dots, \omega_n$ and the modal matrix $\underline{X} = [\bar{x}_1 \dots \bar{x}_n]$ where

$$(-\omega_i^2 \underline{M} + \underline{K})\bar{x}_i = \bar{0}, \quad i = 1, \dots, n \quad (1)$$

and

$$\underline{X}^T \underline{M} \underline{X} = \underline{\mu} = \text{diag}(\mu_1 \dots \mu_n) \text{ and } \underline{X}^T \underline{K} \underline{X} = \underline{\kappa} = \text{diag}(\kappa_1 \dots \kappa_n) \quad (2)$$

where μ_1, \dots, μ_n and $\kappa_1, \dots, \kappa_n$ are constants satisfying $\mu_i > 0$ and $\kappa_i \geq 0$. The kinetic and potential energies of the system are given by

$$T = \frac{1}{2} \dot{\bar{q}}^T \underline{M} \dot{\bar{q}} \text{ and } V = \frac{1}{2} \bar{q}^T \underline{K} \bar{q} \quad (3)$$

respectively. Using *normal coordinates* $\bar{\eta} = (\eta_1 \quad \eta_2 \quad \dots \quad \eta_n)^T$, (see Lecture Notes on p.100) defined by

$$\bar{q} = \underline{X} \bar{\eta} \quad (4)$$

Show that

- a) the kinetic and potential energies may be written

$$T = \frac{1}{2} \dot{\bar{\eta}}^T \underline{\mu} \dot{\bar{\eta}} = \frac{1}{2} \sum_{i=1}^n \mu_i \dot{\eta}_i^2 \text{ and } V = \frac{1}{2} \bar{\eta}^T \underline{\kappa} \bar{\eta} = \frac{1}{2} \sum_{i=1}^n \kappa_i \eta_i^2 \quad (5)$$

- b) the equation for the free motion of the system may be written

$$\ddot{\bar{\eta}} + \underline{\omega}^2 \bar{\eta} = \bar{0}, \text{ where } \underline{\omega}^2 = \underline{\mu}^{-1} \underline{\kappa} \quad (6)$$

Solution:

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