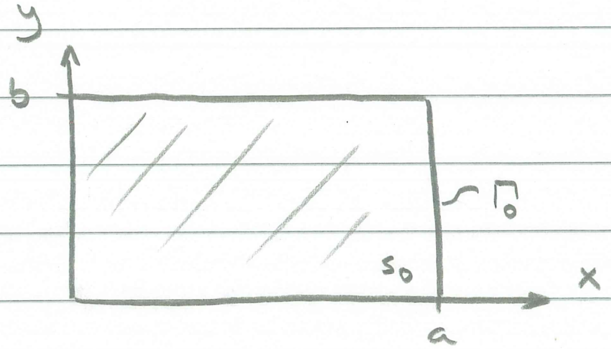
6:5

a)




Simply supported plate.

Free vibrations.

m : Mass per unit area

h : Thickness

E : Modulus of elasticity

$\nu$ : Poisson's ratio

The gouverning equation for thin plates:

$$D \Delta^2 w - p_0 + m_0 \ddot{w} = 0$$ {Page 332 (23.100)}

$$\Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{2\partial^4}{\partial x^2 \partial y^2}$$

(Double Laplacian)

$$D = \frac{Eh^3}{12(1-\nu^2)}$$ Plate bending stiffness {Page 331 (23.99)}

$w = w(x, y, t)$ : Displacement

$p_0$ : External load (Force in the z-direction.)

Free vibrations $\Rightarrow$ $p_0 = 0$

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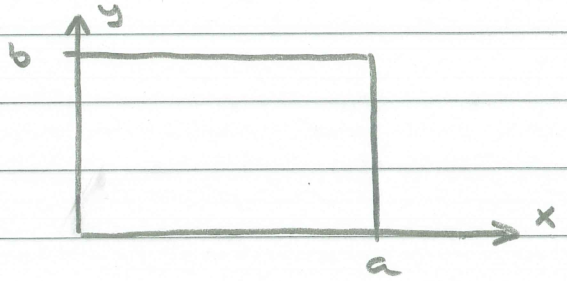
b) Boundary conditions for a simply supported plate:




No displacement on the boundary $\Gamma_0$:

$$w(0,y)=0,\; w(a,y)=0,\; w(x,0)=0,\; w(x,b)=0$$

Simply supported plate $\Rightarrow$ "hinges along the sides".

$$M_x \nu_x^2 + M_y \nu_y^2 + 2 M_{xy} \nu_x \nu_y = 0 \qquad \{\text{Page } 332\ (23.101)\}$$

Constitutive equations for a linear elastic plate:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \qquad \{\text{Page } 331\ (23.98)\}$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

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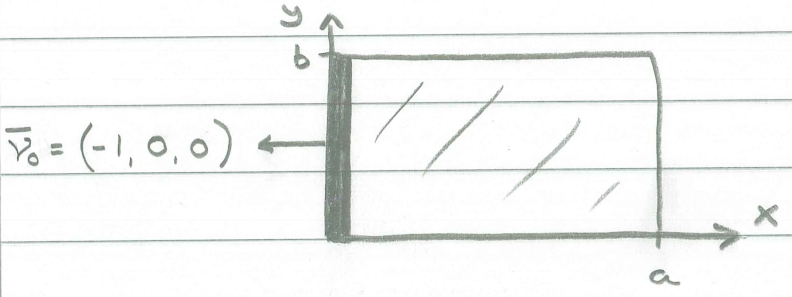
b) • The boundary $(x, y) = (0, y)$

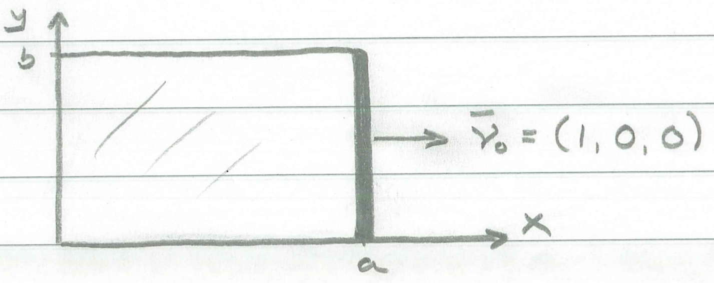



$$M_x \nu_x^2 + M_y \nu_y^2 + 2M_{xy} \nu_x \nu_y = M_x = 0$$

$$M_x(0, y) = -D\left[\frac{\partial^2 w(0,y)}{\partial x^2} + \nu \frac{\partial^2 w(0,y)}{\partial y^2}\right] = -D \frac{\partial^2 w(0,y)}{\partial x^2} = 0$$

since $w(0, y) = 0 \Rightarrow \frac{\partial^2 w(0,y)}{\partial y^2} = 0$

• The boundary $(x, y) = (a, y)$



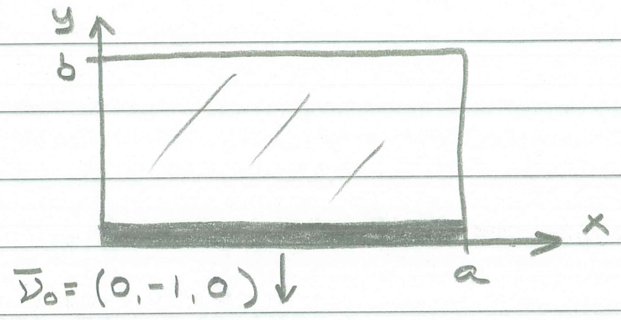

$$M_x \nu_x^2 + M_y \nu_y^2 + 2M_{xy} \nu_x \nu_y = M_x = 0$$

$$M_x(a, y) = -D\left[\frac{\partial^2 w(a,y)}{\partial x^2} + \nu \frac{\partial^2 w(a,y)}{\partial y^2}\right] = -D \frac{\partial^2 w(a,y)}{\partial x^2} = 0$$

since $w(a, y) = 0 \Rightarrow \frac{\partial^2 w(a,y)}{\partial y^2} = 0$

6.5

b) • The boundary $(x,y) = (x,0)$

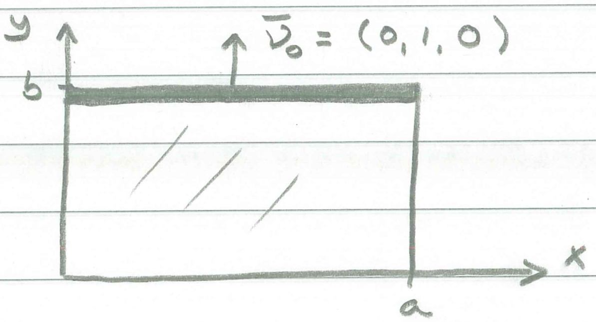



$$M_x \nu_x^2 + M_y \nu_y^2 + 2 M_{xy} \nu_x \nu_y = M_y = 0$$

$$M_y(x,0) = -D\left[\frac{\partial^2 w(x,0)}{\partial y^2} + \nu \frac{\partial^2 w(x,0)}{\partial x^2}\right] = -D \frac{\partial^2 w(x,0)}{\partial y^2} = 0$$

since $w(x,0) = 0 \Rightarrow \dfrac{\partial^2 w(x,0)}{\partial x^2} = 0$

• The boundary $(x,y) = (x,b)$




$$M_x \nu_x^2 + M_y \nu_y^2 + 2 M_{xy} \nu_x \nu_y = M_y = 0$$

$$M_y(x,b) = -D\left[\frac{\partial^2 w(x,b)}{\partial y^2} + \nu \frac{\partial^2 w(x,b)}{\partial x^2}\right] = -D \frac{\partial^2 w(x,b)}{\partial y^2} = 0$$

since $w(x,b) = 0 \Rightarrow \dfrac{\partial^2 w(x,b)}{\partial x^2} = 0$

6.5

c) Look for standing wave solutions and derive the necessary eigen value problems.

Note that no initial values are given, and in part (d) only eigenmodes and corresponding eigenfrequencies are wanted.

Free vibrations, standing wave solution (note that only $\sin\omega t$ is chosen for the time dependent part.)

Ansatz: $w(x,y,t) = \hat{w}(x,y)\sin\omega t$

Insert into the gouverning equation $\Rightarrow$

$$D\Delta^2\hat{w}(x,y)\sin\omega t - m\omega^2\hat{w}(x,t)\sin\omega t = 0$$

$$\Rightarrow \quad \Delta^2\hat{w} - \beta^4\hat{w} = 0 \quad , \quad \beta^4 = \frac{\omega^2 m}{D}$$

Factorize:

$$(\Delta^2 - \beta^4) = (\Delta + \beta^2)(\Delta - \beta^2) = (\Delta - \beta^2)(\Delta + \beta^2)$$

Let $\hat{w}_1$ and $\hat{w}_2$ satisfy

$$(\Delta - \beta^2)\hat{w}_1 = 0 \quad , \quad (\Delta + \beta^2)\hat{w}_2 = 0$$

6.5

c) Then $\hat{w} = \hat{w}_1 + \hat{w}_2$ is a solution for the gouverning equation.

$$(\Delta^2 - \beta^4)\hat{w} = (\Delta + \beta^2)(\Delta - \beta^2)(\hat{w}_1 + \hat{w}_2) = 0$$

General solutions:

Equation $(\Delta - \beta^2)\hat{w}_1 = 0$ has the solution:

$$\hat{w}_1 = Ae^{\alpha x} e^{\gamma y} \quad , \quad \alpha^2 + \gamma^2 = \beta^2$$

Equation $(\Delta + \beta^2)\hat{w}_2 = 0$ has the solution:

$$\hat{w}_2 = Ae^{i\alpha x} e^{i\gamma y} \quad , \quad \alpha^2 + \gamma^2 = \beta^2$$

A general solution to the gouverning equation may then be written:

$$\hat{w}(x,y) = A_1 \sin\alpha x \sin\gamma y + A_2 \cos\alpha x \sin\gamma y +$$

$$+ A_3 \sin\alpha x \cos\gamma y + A_4 \cos\alpha x \cos\gamma y +$$

$$+ A_5 \sinh\alpha x \sinh\gamma y + A_6 \cosh\alpha x \sinh\gamma y +$$

$$+ A_7 \sinh\alpha x \cosh\gamma y + A_8 \cosh\alpha x \cosh\gamma y$$

where boundary conditions determine $A_1, \dots A_8$, $\alpha$ and $\gamma$.

6.5

c) Compare with the general solution to a fourth order differential equation, only dependent on one variable (x).

$$\hat{w}(x) = A_1 \sin \alpha x + A_2 \cos \alpha x +$$

$$+ A_3 \sinh \alpha x + A_4 \cosh \alpha x$$

6.5

d) Boundary conditions.

First consider boundary conditions where either $x = 0$ or $y = 0$.

Note that the boundary conditions are not time dependent and thus they are valid for $\hat{w}$.

$$\hat{w}(0,y) = A_2 \sin\gamma y + A_4 \cos\gamma y + A_6 \sinh\gamma y + A_8 \cosh\gamma y = 0$$

This should be valid for all choises of y. $\Rightarrow$

$$\Rightarrow \quad A_2 = A_4 = A_6 = A_8 = 0$$

$$\hat{w}(x,0) = A_3 \sin\alpha x + A_4 \cos\alpha x + A_7 \sinh\alpha x + A_8 \cosh\alpha x = 0$$

$\left.\begin{array}{l}\text{This should be valid for all choises of } x. \\ (A_4 = A_8 = 0 \text{ is shown above.})\end{array}\right\} \Rightarrow$

$$\Rightarrow \quad A_3 = A_7 = 0$$

Thus

$$\hat{w}(x,y) = A_1 \sin\alpha x \sin\gamma y + A_5 \sinh\alpha x \sinh\gamma y$$

6.5

d)

$$\hat{w}(x,y) = A_1 \sin\alpha x \sin\gamma y + A_5 \sinh\alpha x \sinh\gamma y$$

$$\frac{\partial^2 \hat{w}(x,y)}{\partial x^2} = -A_1 \alpha^2 \sin\alpha x \sin\gamma y + A_5 \alpha^2 \sinh\alpha x \sinh\gamma y$$

$$\frac{\partial^2 \hat{w}(x,y)}{\partial y^2} = -A_1 \gamma^2 \sin\alpha x \sin\gamma y + A_5 \gamma^2 \sinh\alpha x \sinh\gamma y$$

$$\frac{\partial^2 \hat{w}}{\partial x^2}(a,y) = -A_1 \alpha^2 \sin\alpha a \sin\gamma y + A_5 \alpha^2 \sinh\alpha a \sinh\gamma y = 0$$

This should be valid for all choises of $y$. $\Rightarrow$

1) $-A_1 \alpha^2 \sin\alpha a = 0$ , $\alpha \neq 0$ $\Rightarrow$ $\sin\alpha a = 0$

2) $A_5 \alpha^2 \sinh\alpha a = 0$ , $\alpha \neq 0$ $\Rightarrow$ $A_5 = 0$

($\sinh x = 0$ for $x = 0$ only)

1) $\Rightarrow$ $\sin\alpha a = 0$ , $\alpha = \alpha_m = \frac{m\pi}{a}$ , $m = 1, 2, \ldots$

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d)

$$\frac{\partial^2 \hat{w}}{\partial y^2}(x,b) = -A_1 \gamma^2 \sin\alpha x \sin\gamma b + A_5 \gamma^2 \sinh\alpha x \sinh\gamma b = 0$$

This should be valid for all choises of $x$. $\Rightarrow$

3) $-A_1\gamma^2 \sin\gamma b = 0$ , $\gamma \neq 0$ $\Rightarrow$ $\sin\gamma b = 0$

4) $A_5\gamma^2 \sinh\gamma b = 0$ , $\gamma \neq 0$ $\Rightarrow$ $A_5 = 0$

($\sinh x = 0$ for $x = 0$ only)

3) $\Rightarrow$ $\sin\gamma b = 0$ , $\gamma = \gamma_n = \frac{n\pi}{b}$ , $n = 1, 2, \ldots$

With $\beta^2 = \beta_{mn}^2 = \alpha_m^2 + \gamma_n^2 = \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$

and $\beta^4 = \frac{\omega^2 m}{D}$

$\Rightarrow$ $\omega = \omega_{mn} = \beta_{mn}^2\sqrt{\frac{D}{m}} = \pi^2\sqrt{\frac{D}{m}}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]$ : Eigenfrequencies

and corresponding eigenmodes (mode shapes):

$$\hat{w}_{mn}(x,y) = A_1 \sin\alpha_m x \sin\gamma_n x = A_1 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

$m, n = 1, 2, \ldots$