7. Boundary-value problems

***Problem statement in CM* consists of three distinct parts:**

(1) the partial differential *field equations* to be satisfied

(2) the *unknown fields* of the problem and the relations between them

(3) the *prescribed data*, which include everything else that is required to turn the problem into one that *can* be solved

Initial boundary-value – the three parts will all have a temporal component and we are interested in the dynamic response of a system

boundary-value problem - If we are only interested in the static equilibrium state of our system

7. Boundary-value problems

In addition to the above considerations, continuum mechanics problems naturally divide into two further categories

1. those which are ***formulated within the spatial description*** (useful for fluid mechanics problems)
2. those that are ***formulated within the material description*** (useful for solid mechanics problems)

We focus on pure mechanical behavior of materials (ignore thermodynamics)

7.1.1. Problems in the spatial description – Eulerian approach

The first part of a problem is the field equations which, in this case, are the continuity equation (conservation of mass, Eqn. (4.3)) and the balance of linear momentum (Eqn. (4.25)).

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0, \\ &\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho (\nabla \boldsymbol{v}) \boldsymbol{v}, \end{aligned}\right\} \qquad \boldsymbol{x} \in B,\ t > 0$$

In addition the balance of angular momentum leading to the symmetry of the Cauchy stress tensor is imposed.

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

7.1.1. Problems in the spatial description – Eulerian approach

The second part of a problem is the set of unknown fields. The unknowns are taken to be the density field $\rho(\boldsymbol{x}, t)$ and the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. The aim of the problem is to determine these fields

The third part of a problem are the prescribed data, which includes the *initial conditions* and *boundary conditions* for the unknown fields. We also need a specification of functions that provide the body forces and the Cauchy stress.

initial conditions : $\rho(\boldsymbol{x}, 0) = \rho_{\text{init}}(\boldsymbol{x})$ $\qquad$ $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_{\text{init}}(\boldsymbol{x})$

boundary conditions : $\boldsymbol{v}(\boldsymbol{x}, t) = \bar{\boldsymbol{v}}(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \partial B$

Another possibility is to impose *traction boundary conditions:*

$$\boldsymbol{\sigma n}(\boldsymbol{x}, t) = \bar{\boldsymbol{t}}(\boldsymbol{x}, t), \qquad \boldsymbol{x} \in \partial B,\ t > 0$$

7.1.1. Problems in the spatial description – Eulerian approach

If the problem is one of ***steady state***, this means that the time derivatives in Eqn. (7.1) are zero and so the set of differential equations reduces to

$$\left.\begin{aligned} &\operatorname{div} \rho \boldsymbol{v} = 0, \\ &\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho(\nabla \boldsymbol{v})\boldsymbol{v}, \end{aligned}\right\} \qquad \boldsymbol{x} \in B$$

called *steady-state stress equations*.

In this case initial conditions are not needed. Everything is independent of time and only the (constant) boundary conditions must be specified.

7.1.2. Problems in the material description – Lagrangian description

The continuum mechanics initial boundary-value problem can also be formulated in the material description. This is referred to as a Lagrangian description.

The first part of a problem is the field equations. In the material description, the balance of linear momentum is given terms of the first Piola–Kirchhoff stress or in terms of the second Piola–Kirchhoff stress.

Usually boundary value problems are solved numerically by discritezation (FEM). This is not the aim of this course – we look only for analytical solutions.

10. Approximate solutions: Reduction to the engineering theories

Continuum mechanics is in many ways the "***grand unified theory***" of engineering science. The governing equations of continuum mechanics provide the most general description of the behavior of materials (solid and fluid) under arbitrary loading.

Any engineering problem can therefore be described as a solution to the following coupled system of equations (together with the appropriate constitutive relations and initial and/or boundary conditions) :

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0,$$

balance of mass

$$\operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v}) \boldsymbol{v} \right],$$

balance of linear momentum

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T,$$

balance of angular momentum

$$\boldsymbol{\sigma} : \boldsymbol{d} + \rho r - \operatorname{div} \boldsymbol{q} = \rho \left[\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u \right],$$

conservation of energy

10. Approximate solutions: Reduction to the engineering theories

Due to the nonlinearity (material and geometric) of the resulting initial/boundary-value problem, analytical solutions are unavailable except in very few cases.

This leaves two options. Either a numerical solution must be pursued or the governing equations and/or constitutive relations must be simplified.

We discuss various simplifications of the continuum equations that lead to more approximate theories that nevertheless provide great insight into physical behavior.

10. Approximate solutions: Reduction to the engineering theories

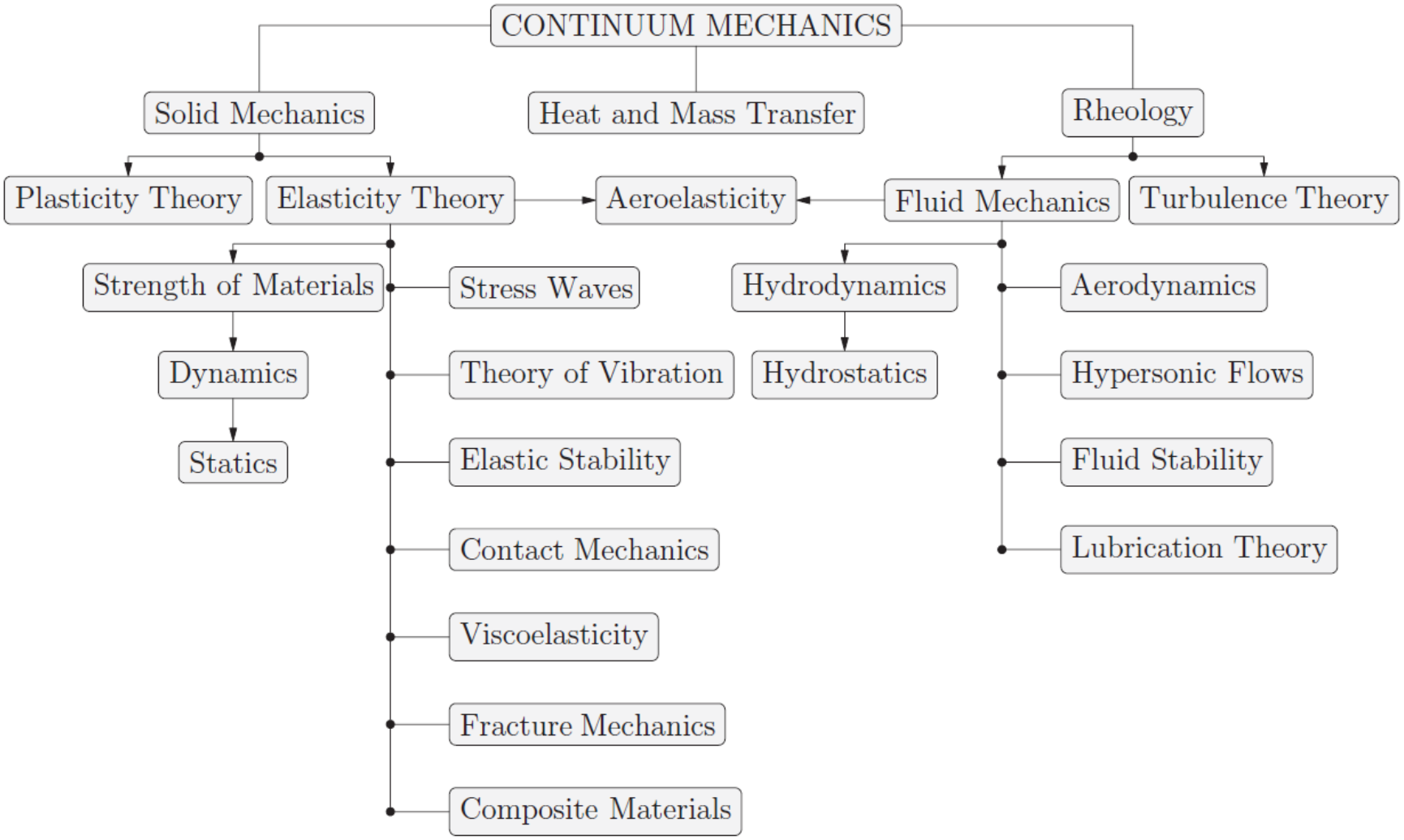

Fig. 10.1 Continuum mechanics as the "grand unified theory" of engineering science.

10.1 Mass transfer theory

- Start with the continuity equation:

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \boldsymbol{v}) = 0$$

- Define the *mass flux vector:*

$$\boldsymbol{j} \equiv \rho \boldsymbol{v}$$

- Make the constitutive assumption, referred to as *Fick's law*

$$\boldsymbol{j} = -\widehat{D}(\rho)\nabla\rho(\boldsymbol{x}, t)$$

$D = \widehat{D}(\rho)$ - *diffusion coefficient*

- Substituting Fick's law into the continuity equation

$$\frac{\partial \rho}{\partial t} - \text{div}\left[\widehat{D}(\rho)\nabla\rho\right] = 0$$

nonlinear diffusion equation

- If *D* is a constant, the result is the *linear diffusion equation*

$$\frac{\partial \rho}{\partial t} = D\nabla^2\rho$$

10.2 Heat transfer theory

Focuses entirely on the transfer of energy via heat. Energy flux due to mechanical work is neglected.

The energy equation is then an independent equation and reduces to:

$$\rho r - \operatorname{div} \boldsymbol{q} = \rho \frac{\partial u}{\partial t}$$

We add to this two constitutive postulates:

1. The local form of ***Joule's law*** $u = u_0 + c_v T$

c_v - *specific heat capacity*

u_0 - reference internal energy density

10.2 Heat transfer theory

2. Fourier's law $\boldsymbol{q} = -k\nabla T$

k - *thermal conductivity* of the material.

Substituting the two constitutive laws into the energy equation:

$$\rho r + k\nabla^2 T = \rho c_v \frac{\partial T}{\partial t}$$

In the absence of internal heat sources (r = 0) the equation reduces to

$$k\nabla^2 T = \rho c_v \frac{\partial T}{\partial t}$$

Note that it has the same mathematical form as the diffusion equation, although physically the equations describe different phenomena.

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho$$

10.3 Fluid mechanics theory

The basic theory of fluid mechanics deals with the flow of Newtonian fluids:

$$\boldsymbol{\sigma} = -p(\rho)\boldsymbol{I} + \left[\kappa(\rho) - \frac{2}{3}\mu(\rho)\right](\operatorname{tr}\boldsymbol{d})\boldsymbol{I} + 2\mu(\rho)\boldsymbol{d}$$

κ and μ are the bulk and shear viscosities

Substituting this relation into the balance of linear momentum gives:

$$\underbrace{-\nabla p}_{\text{pressure gradient force}} + \underbrace{\nabla\left[\left(\kappa - \frac{2}{3}\mu\right)\operatorname{tr}\boldsymbol{d}\right] + 2\operatorname{div}(\mu\boldsymbol{d})}_{\text{viscous forces}} + \underbrace{\rho\boldsymbol{b}}_{\text{body forces}} = \rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla\boldsymbol{v})\boldsymbol{v}\right]$$

The generalized ***Navier–Stokes equations*** can describe the most general kinds of *laminar* flows, i.e. flows in which the fluid elements move in parallel layers, that Newtonian fluids can undergo.

10.3 Fluid mechanics theory

The Navier–Stokes equations are often simplified by making some additional approximations.

Assuming incompressible flow for which div ***v*** = 0

$$-\nabla p + \mu\nabla^2 \boldsymbol{v} + \rho \boldsymbol{b} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\boldsymbol{v} \right]$$

This is the form of the Navier–Stokes equations that is most familiar to engineers and is used most often in practical applications.

For an ideal nonviscous fluid, $\mu = 0$, and we obtain the ***Euler equation***

$$-\nabla p + \rho \boldsymbol{b} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\boldsymbol{v} \right]$$

10.3 Fluid mechanics theory

$$-\nabla p + \rho \boldsymbol{b} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\boldsymbol{v} \right]$$

Finally, in the static case $(\boldsymbol{v} = \boldsymbol{0})$, we obtain the hydrostatic equations:

$$\nabla p = \rho \boldsymbol{b}$$

which describe the behavior of a stationary fluid subjected to body forces.

10.4 Elasticity theory - restricted to linear elastic materials

Further we assume that the displacement gradients
are small relative to unity, so that the Lagrangian strain tensor

$$\boldsymbol{E} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T + (\nabla \boldsymbol{u})^T \nabla \boldsymbol{u}\right]$$

$\boldsymbol{u}$ is the displacement field

can be approximated by the small-strain tensor

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right]$$

10.4 Elasticity theory

The appropriate constitutive relation for this case is the generalized Hooke's law

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} = c_{ijkl}u_{k,l}$$

c_{ijkl} - elasticity tensor

Substituting Hooke's law into the balance of linear momentum:

$$(c_{ijkl}u_{k,l})_{,j} + \rho b_i = \rho\frac{\partial^2 u_i}{\partial t^2}$$

Navier equations for a linear elastic solid

10.4 Elasticity theory

For the simplest case of a homogeneous isotropic material the Navier equations take the form:

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) + \rho\boldsymbol{b} = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}$$

Navier equations are linear and for this reason closed-form solutions for elasticity problems are much easier to find than those for fluid mechanics.

Special case: ***static boundary-value problems*** for which the Navier equations reduce to

$$\mu\nabla^2\boldsymbol{u} + (\lambda+\mu)\nabla(\operatorname{div}\boldsymbol{u}) + \rho\boldsymbol{b} = \boldsymbol{0}$$

Further simplification is possible by restricting the equations to two dimensions and making certain kinematic. If the body is very thin in the third direction, ***plane stress*** conditions are assumed to hold. Conversely, if the body is “infinite” in the third direction, ***plane strain*** conditions are assumed.