Multibody Dynamics
Spring 2019

www.mek.lth.se
Education/Advanced Courses

MULTIBODY DYNAMICS (FLERKROPPSDYNAMIK)
(FMEN02, 7.5p)
Course coordinator

• Prof. Aylin Ahadi, Mechanics, LTH
• aylin.ahadi@mek.lth.se
• 046-2223039
# COURSE REGISTRATION

**Multibody Dynamics (FMEN02) Spring 2018**

<table>
<thead>
<tr>
<th>pnr</th>
<th>efternamn</th>
<th>förnamn</th>
<th>email</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Per Lidström  Kristina Nilsson

Division of Mechanics, Lund University
Password: euler

Leonard Euler 1707-1783
Course program


Teacher: Prof. Aylin Ahadi (Lectures, Exercises, Course coordination)

Lectures: Monday 13 - 15  Tuesday 13 - 15  Thursday 10 - 12, room M:IP2

Exercises: Wednesday 13 - 15, room M:IP2

<table>
<thead>
<tr>
<th>Term</th>
<th>Monday 25/3</th>
<th>Tuesday 26/3</th>
<th>Thursday 27/3</th>
<th>Thursday 28/3</th>
<th>Friday 29/3</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10:00</td>
<td></td>
<td>10:00</td>
<td>10:00</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FMEN02</td>
<td></td>
<td>FMEN02</td>
<td>FMEN02</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Förel</td>
<td></td>
<td>Förel</td>
<td>Förel</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Flerkropps dynamik</td>
<td></td>
<td>Flerkropps dynamik</td>
<td>Flerkropps dynamik</td>
<td></td>
</tr>
<tr>
<td></td>
<td>BME4, F4-bem, F4-tf, M4-bem, Pi4-bem</td>
<td></td>
<td>BME4, F4-bem, F4-tf, M4-bem, Pi4-bem</td>
<td>BME4, F4-bem, F4-tf, M4-bem, Pi4-bem</td>
<td></td>
</tr>
</tbody>
</table>
Course program


Teacher: Prof. Aylin Ahadi (Lectures, Exercises, Course coordination)
Phone: 046-222 3039, email: aylin.ahadi@mek.lth.se

Lectures: Monday 13 - 15  Tuesday 13 - 15  Thursday 10 - 12, room M:IP2
Exercises: Wednesday 13 - 15, room M:IP2

<table>
<thead>
<tr>
<th>v/t</th>
<th>Må 1/4</th>
<th>Ti 2/4</th>
<th>On 3/4</th>
<th>To 4/4</th>
<th>Fr 5/4</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### w.1 (March 25-29)  **Particle dynamics**

*Chapter 3: One particle systems* and *Many particle systems* are left to the student for *self study*. Theses chapters give a recapitulation of concepts and definitions on the level of a basic course in mechanics.

**Exercise 0:**

11, 15, 16

### Rigid body kinematics

**Lecture 1:**

Introduction: Course objectives. Examination procedures.

Multibody Dynamics: Introduction. Rigid body kinematics.

**Lecture 2:**

Rigid body kinematics: Euler’s theorem. Screw displacement.

Mathematical representation of rotations: The main representation theorem. Euler’s angles. Euler parameters.

**Lecture 3:**

Euler’s angles. Euler parameters. Angular velocity and acceleration. Euler-Poisson’s velocity formula.

### w.2 (April 1-5)  **Rigid body dynamics**

**Exercise 1:**

1, 7, 9, 11, 12, 13a-d, 14, 19, 20

**Lecture 4:**

Equations of motion: Newton’s and Euler’s equations of motion.

Internal forces. Equations of balance. Power and energy.

**Exercise 2a:**

12, 13, 17, 20, 21, 22

**2b:**

1, 4, 7, 9, 14, 15

**Lecture 5:**

Rigid body dynamics: Equations of motion. Inertia tensor.

Principal axes. Free axes. Symmetries. Euler’s equations:

The moment-free motion. Stability. The ADAMS software.

### w3 (April 8-12)  **The Deformable body**

**Lecture 6:**


**Lecture 7:**

Stress tensors. Power and energy. The elastic body.

The Green-St Venant material. The elastic potential.

**Exercise 3:**

4, 7, 9, 10, 12, 16, 21, 22a

**Multibody kinematics and dynamics**

**Lecture 8:**

<table>
<thead>
<tr>
<th>Week</th>
<th>Topic</th>
<th>Lectures</th>
<th>Exercises</th>
</tr>
</thead>
<tbody>
<tr>
<td>w.4</td>
<td>Multibody kinematics and dynamics</td>
<td>Lecture 9: Lagrange’s equations. Kinetic energy. Generalized forces.</td>
<td>9.4, 9.4.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exercise 4: 1, 2, 6, 7, 12, 13</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Tuesday 10-12</td>
<td></td>
</tr>
<tr>
<td>w.5</td>
<td>Multibody dynamics</td>
<td>Lecture 10: Regular coordinates and coordinate changes. The rigid part.</td>
<td>9.4.2, 9.5,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>The elastic part. External forces.</td>
<td>9.6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Lecture 11: The interaction between parts. The interaction between rigid</td>
<td>9.7</td>
</tr>
<tr>
<td></td>
<td></td>
<td>parts.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exercise 5: 2, 3, 5, 6, 8</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Lecture 12: Constraints (Holonomic and non-holonomic). The power theorem.</td>
<td>9.7.1, 9.8</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>w.6</td>
<td>Constrained Multibody dynamics</td>
<td>Lecture 13: Constraint conditions. Lagrange’s equations with constraint</td>
<td>10.1, 10.2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>conditions. Lagrangian multipliers. Examples.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Lecture 14: Examples continued. The power theorem. Application project</td>
<td>10.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>specification. The elastic beam.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exercise 6a: 2, 3, 4, 5, 9, 10</td>
<td></td>
</tr>
<tr>
<td>w.7</td>
<td>Coordinate representations</td>
<td>Lecture 15: Project work with ADAMS, Help desk and application</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Lecture 16: Recap and previous written tests</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exercise 6b: 1, 6, 7, 8, 11</td>
<td></td>
</tr>
</tbody>
</table>
Course objectives and contents

Course objectives:
The objective of this course is to present the basic theoretical knowledge of the Foundations of Multibody Dynamics with applications to machine and structural dynamics. The course gives a mechanical background for applications in, for instance, control theory and vehicle dynamics.

Course contents:
1) Topics presented at lectures and in the “Lecture Notes”.
2) Exercises.
3) Examination tasks
4) Application project
The scope of the course is defined by the curriculum above and the lecture notes (Fundamentals of Multibody Dynamics, Lecture Notes).

The teaching consists of Lectures and Exercises:
Lectures: Lectures will present the topics of the course in accordance with the curriculum presented above.
Exercises: Recommended exercises worked out by the student will serve as a preparation for the Examination tasks. Solutions to selected problems will be distributed.
Examination

Examination:
The examination consists of two parts:

1. Hand in solutions to 3 examination tasks (assignments containing 2-4 problems each) and one Application project. The examination tasks will be handed out starting by w.2. Last date for handing in solution to the examination tasks and the project report:

   Examination task 1: Rigid body kinematics          April 12th
   Examination task 2: Rigid and flexible body dynamics May 17th
   Examination task 3: Multibody dynamics            May 24th
   Application project (with Adams software)          May 31th

2. Written examination containing five (5) tasks focusing on the theoretical parts of the course content. A correctly solved task gives 3p (Total possible score 5x3p = 15p).

   Date for the written examination: Monday, June 3rd, 8:00-13:00 in Vic: 2C

Credit requirements:
Credit 3: Solutions to all examination tasks with 65% of the problems correctly solved. Approved project report. Written examination with 50% of the tasks correctly solved (7.5p).
Credit 4: Solutions to all examination tasks with 75% of the problems correctly solved. Approved project report. Written examination with 67% of the tasks correctly solved (10p).
Credit 5: Solutions to all examination tasks with 90% of the problems correctly solved. Approved project report. Written examination with 83% of the tasks correctly solved (12.5p).
# CONTENTS

1. INTRODUCTION ........................................ 1
2. BASIC NOTATIONS .................................... 12
3. PARTICLE DYNAMICS .................................. 13
   3.1 One particle system ................................ 13
   3.2 Many particle system .............................. 27
4. RIGID BODY KINEMATICS ............................. 36
   4.1 The rigid body transplacement .................... 36
   4.2 The Euler theorem .................................. 47
   4.3 Mathematical representations of rotations ....... 59
   4.4 Velocity and acceleration ......................... 72
5. THE EQUATIONS OF MOTION ......................... 89
   5.1 The Newton and Euler equations of motion ...... 89
   5.2 Balance laws of momentum and moment of momentum 94
   5.3 The relative moment of momentum ............... 99
   5.4 Power and energy .................................. 100
6. RIGID BODY DYNAMICS ............................... 104
   6.1 The equations of motion for the rigid body ..... 104
   6.2 The inertia tensor .................................. 109
   6.3 Fixed axis rotation and bearing reactions ....... 138
   6.4 The Euler equations for a rigid body .......... 143
   6.5 Power, kinetic energy and stability .............. 158
   6.6 Solutions to the equations of motion: An introduction to the case of 172
7. THE DEFORMABLE BODY ................................ 172
   7.1 Kinematics ......................................... 172
   7.2 Equations of motion ................................ 196
   7.3 Power and energy .................................. 205
   7.4 The elastic body ................................... 209
8. THE PRINCIPLE OF VIRTUAL POWER ............... 217
   8.1 The principle of virtual power in continuum mechanics 219
# Mechanics in perspective

<table>
<thead>
<tr>
<th>Quantum</th>
<th>Nano</th>
<th>Micro-meso - Macro-Continuum</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
</tbody>
</table>

**lengthscale**

Å  nm  µm  mm
Prerequisites

Multibody Dynamics

Synthesis

- 3D Analysis
- Linear algebra

- Continuum mechanics
- Mechanics (Basic course)

- Finite element method
- Matlab

Mathematics
Mechanics
Numerical analysis
A multibody system is a mechanical system consisting of a number of interconnected components, or parts, performing motions that may involve large translations and rotations as well as small displacements such as vibrations.

Interconnections between the components are of vital importance. They will introduce constraints on the relative motion between components and in this way limit the possible motions which a multibody system may undertake.
Course objectives

Primary objectives are to obtain

a thorough understanding of

- rigid body dynamics
- Lagrangian techniques for multibody systems containing constraint conditions

a good understanding of

- the dynamics multibodies consisting of coupled rigid bodies

some understanding of

- the dynamics of multibodies containing flexible structures
an ability to

♠ perform an analysis of a multibody system and to present the result in a written report

♠ read technical and scientific reports and articles on multibody dynamics

some knowledge of

♠ industrial applications of multibody dynamics

♠ computer softwares for multibody problems
Industrial applications

Multibody system dynamics (MBS dynamics) is motivated by an increasing need for analysis, simulations and assessments of the behaviour of machine systems during the product design process.
What is a multibody system?

A **multibody system** is a mechanical system consisting of a number of interconnected components, or *parts*, performing motions that may involve large *translations* and *rotations* as well as small displacements such as *vibrations*. This is often the case for machines produced in the automotive, aerospace and robotic industries.
What is a multibody system?

The *interconnections* between the components are of vital importance. They will introduce *constraints* on the relative motion between components and in this way *limit the possible motions* which a multibody system may undertake.
Commercial MBS softwares

- Adams
- Dads
- Simpac
- Ansys
- Dymola/Modelinc
- Simulink/Matlab
- RecurDyn

Adams Student Edition
Multibody Dynamics Simulation

https://www.youtube.com/watch?v=hd-cFu7yMGw
Multibody system

Classical steam engine equipped with a *slider crank mechanism*

*Slider crank mechanism* converts reciprocating translational motion of the slider into rotational motion of the crank or the other way around.
The **crank** is connected to the **slider** via a **rod** which performs a translational as well as rotational motion. The connection between the slider and the rod is maintained through a so-called **revolute joint** and a similar arrangement connects the rod and the crank.
Multibody system

A schematic (topological) picture of the slider crank mechanism. The mechanism consists of three bodies; the slider, the crank and the rod. These are connected by revolute joints which allow them to rotate relative one another in plane motion. It is practical to introduce a fourth part, the ground, which is assumed to be resting in the frame of reference.
The **revolute joint** works like a hinge and allows two connected parts to rotate relative to one another around an axis, fixed in both parts.

A **prismatic or translational joint** allows two connected parts to translate relative to one another along an axis, fixed in both parts.

The position of the slider can be measured by the single coordinate. We say that the prismatic joint has one *degree of freedom*.
Configuration coordinates
Rigid part in plane motion, 3DOF

Configuration coordinates: $\theta, x_C, y_C$
Figure 1.3 Topology and coordinates of slider crank mechanism.

# Parts: 4

Configuration coordinates: \(x_B, \theta, \varphi, x_C, y_C\)
Multibody system

Configuration coordinates:

$x_B, \theta, \phi, x_C, y_C$

Due to the constraints upheld by the revolute joints at A and B, the number of degrees of freedom for the system is less than five.

A revolute joint removes two degrees of freedom from the multibody system. Two revolute joints for this case, four degrees of freedom are removed and we are left with one.

In the analysis of a multibody system, in plane motion, one may use the Greubler formula which gives the number of degrees of freedom.
Greubler formula
Two dimensional

\[ f = 3(N - 1) - \sum_{i=1}^{m} r_i \]

Number of degrees of freedom (DOF): \( f \)

Number of parts: \( N \) (including the ground)

Number of joints: \( m \)

Number of DOF removed by joint \( i \): \( r_i \)

Revolute joint: \( r_i = 2 \)  
Translational joint: \( r_i = 2 \)
Joints in three dimensions

**Figure 1.3** The revolute joint.

\[ r_i = 5 \]

**Figure 1.4** The prismatic joint.

\[ r_i = 5 \]

\[ f = 6(N - I) - \sum_{i=1}^{m} r_i \]
Degrees of freedom and constraints

Figure 1.5  a) Three-bar mechanism  b) The double pendulum.
Greubler formula

\[ f = 3(N-1) - \sum_{i=1}^{m} r_i, \quad f \geq 1 \]

Slider crank mechanism:

\[ N = 4, \quad m = 4, \quad r_i = 2 \]

\[ f = 3(4-1) - 4 \cdot 2 = 1 \]

Three bar mechanism: \( N = m = 4, \quad r_i = 2 \)

\[ f = 3(4-1) - 4 \cdot 2 = 1 \]

Double pendulum: \( N = 3, \quad m = 2, \quad r_i = 2 \)

\[ f = 3(3-1) - 2 \cdot 2 = 2 \]
Biomechanics

A simple multibody model of a humanoid in plane locomotion

Figure 1.6 Human locomotion.

\[ N = 11, \quad m = 10, \quad r_i = 2 \quad \Rightarrow \quad f = 3(11 - 1) - 10 \cdot 2 = 30 - 20 = 10 \]
If the number of configuration coordinates is equal to the number of degrees of freedom of the system then we say that we use a **minimal set of coordinates**. Solutions to these equations represent the motion of the multibody system.

Using this approach will hide important information on the dynamics of the system. For instance, **internal forces** in the system, such as forces appearing in the joints, will not be accessible using a minimal set of coordinates.
To calculate these forces one has to increase the number of configuration coordinates and along with this introduce constraint conditions relating these coordinates in accordance with the properties of the joints between the parts.

Configuration coordinates: \( x_B, \theta, \varphi, x_C, y_C \)

Constraint at A:

\[
\begin{align*}
-R \cos \theta + \frac{L}{2} \cos \varphi - x_C &= 0 \\
R \sin \theta - \frac{L}{2} \sin \varphi - y_C &= 0
\end{align*}
\]

Constraint at B:

\[
\begin{align*}
x_B - \frac{L}{2} \cos \varphi - x_C &= 0 \\
\frac{L}{2} \sin \varphi - y_C &= 0
\end{align*}
\]
Multibody dynamics - the analysis of a mechanical system results in a solution of a set of ordinary differential equations

• To set up the differential equations we need a geometrical description of the MBS and its parts including its mass distribution. (In the case of flexible parts we also need to constitute the elastic material properties of these parts.)

• We have to analyze the connections between parts in terms of their mechanical properties. Properties of the connections in the shape of joints will be represented by constraint conditions, i.e. equations relating the configuration coordinates and their time derivatives.

• Finally the interaction between the MBS and its environment has to be defined and given a mathematical representation. Based on the fundamental principles of mechanics; Euler’s laws and the Principle of virtual power we may then formulate the differential equations governing the motion of the MBS.

• The solution of these equations will, in general, call for numerical methods.
Multibody dynamics software Adams

Figure 1.7 Examples of MBS models in ADAMS®.
The rigid body does not change its shape during the motion.

This model concept may be considered to be the most important for MBS and many MBS are equal to systems of coupled rigid bodies.

The general motion of a rigid body thus has six degrees of freedom.

\[
\begin{align*}
  v_P &= v_c + \omega \times p_{cP} \\
  a_P &= a_c + \alpha \times p_{cP} + \omega \times (\omega \times p_{cP}), \quad P \in \mathcal{B}
\end{align*}
\]

The equations of motion

\[
\begin{align*}
  F^e &= a_c m \\
  M^e_c &= I_c \dot{\omega} + \omega \times I_c \omega
\end{align*}
\]

\( I_c \) is the inertia tensor of the body with respect to its centre of mass.
The flexible body

Figure 1.9 The flexible body.

The elastic energy

\[ U_e = \int_{B_0} \left( \frac{1}{2} \lambda (\text{tr}E)^2 + \mu |E|^2 \right) dv(X) \]

Green-St.Venant strain tensor and Lame-moduli
The Multibody kinematics

Configuration coordinates:

\[ q = q^1, q^2, \ldots, q^n \]

expressed in terms of these coordinates and their time derivatives

Kinetic energy: \( T = T(q, \dot{q}) \) \quad Potential energy: \( U = U(q) \)

Equations of motion: (Lagrange’s equations)

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} + \frac{\partial U}{\partial q_k} - Q_k = 0, \quad k = 1, \ldots, n
\]

\[ Q_k = Q_k(q, \dot{q}) \quad \text{Generalized force} \] represents external forces on the system and internal contact forces between parts
Equations of motion under constraints

Lagrangian: \( L = T - U \)

Nonconservative forces: \( Q_{k}^{\text{non}} = Q_{k}^{\text{non}}(t, q, \dot{q}) \)  

Constraint equations: \( l \leq m < n \)

\[ g_{v0} + \sum_{k=1}^{n} g_{vk} \dot{q}^k = 0, \quad \nu = 1, \ldots, m \quad \text{with} \quad g_{vk} = g_{vk}(t, q) \]

Equations of motion:

\[
\begin{cases}
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^k} \right) - \frac{\partial L}{\partial q^k} - Q_{k}^{\text{non}} - \sum_{\nu=1}^{m} \lambda^\nu g_{vk} = 0, \quad k = 1, \ldots, n \\
 g_{v0} + \sum_{k=1}^{n} g_{vk} \dot{q}^k = 0, \quad \nu = 1, \ldots, m
\end{cases}
\]

\( Q_{k}^{\text{ke}} = \sum_{\nu=1}^{m} \lambda^\nu g_{vk}, \quad k = 1, \ldots, n \)  

reaction forces due to the kinematical constraints

Constraint forces: \( \lambda^\nu = \lambda^\nu(t), \quad \nu = 1, \ldots, m \)
Chapter 3: Particle Dynamics

is left for **self study** and **recapitulation** of basic concept i mechanics.

You may try to solve some of the exercises but **don’t spend to much time** on this introductory chapter!
Euclidean space

\[ x, y, z \ldots \in \mathbb{E} \quad \text{Points in Euclidean space} \]

\[ y - x = \mathbf{u} \in \mathcal{V} \quad \text{Vectors in Translation vector space} \]

\[ y = x + \mathbf{u} \]

\[ \dim(\mathcal{V}) = 3 \quad \mathbf{e} = (e_1, e_2, e_3) \quad \text{basis} \]

\[ \mathbf{a} = e_1 a_1 + e_2 a_2 + e_3 a_3 = (e_1, e_2, e_3) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \mathbf{e} [\mathbf{a}]_\mathbb{E} \]
Cartesian coordinate system

\[ o \in \mathcal{E} \quad \text{Fixed point,} \quad \underline{e} = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix} \quad \text{Fixed basis} \]

\[ x = o + r_{ox} = o + e_1 x_1 + e_2 x_2 + e_3 x_3 = o + \underline{e}[r_{ox}]_e = o + \underline{e}[x]_e \]

\[ o' \in \mathcal{E} \quad \text{Fixed point,} \quad \underline{f} = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} \quad \text{Fixed basis} \]

\[ x = o' + \underline{f}[x]_f \quad \quad o' = o + \underline{e}[o']_e \]

\[ x = o + \underline{e}[o']_e + \underline{f}[x]_f = o + \underline{e}[x]_e \Rightarrow \underline{f}[x]_f = \underline{e}([x]_e - [o']_e) \]

\[ \underline{f} = \underline{e}B, \quad B \in \mathbb{R}^{3 \times 3}, \quad \text{det} \, B \neq 0 \]
**Cartesian coordinate system**

\[ eB[x]_f = e([x]_e - [o']_e) \Leftrightarrow B[x]_f = [x]_e - [o']_e \]

**Transformation of coordinates for points**

\[ [x]_f = B^{-1}([x]_e - [o']_e) \]

\[ [x]_e = B([x]_f - [o]_f) \]
What about vectors?

\[ a \in V \]

\[ a = e[a]_e, \quad a = f[a]_f, \quad f = eB \]

\[ a = f[a]_f = eB[a]_f = e[a]_e \iff [a]_e = B[a]_f \]

\[ [a]_e = B[a]_f, \quad [a]_f = B^{-1}[a]_e \]
What about tensors?

\[ A \in \text{End}(\mathcal{V}) \]

\[ Ae = e[A]_e, \quad Af = f[A]_f, \quad f = eB \]

\[
\begin{align*}
[A]_f &= B^{-1}[A]_e B \\
[A]_e &= B[A]_f B^{-1}
\end{align*}
\]

\[ B = e^T f \]
Orthonormal basis

\[ e = (e_1, e_2, e_3) \]

\[ e_i \cdot e_j = \delta_{ij} = \begin{cases} 
1 & i = j \\
0 & i \neq j 
\end{cases} \]

\[
\begin{pmatrix}
  (e_1) \\
  (e_2) \\
  (e_3)
\end{pmatrix}
\cdot
\begin{pmatrix}
  (e_1) & (e_2) & (e_3)
\end{pmatrix}
=
\begin{pmatrix}
  (e_1 \cdot e_1) & (e_1 \cdot e_2) & (e_1 \cdot e_3) \\
  (e_2 \cdot e_2) & (e_2 \cdot e_2) & (e_2 \cdot e_3) \\
  (e_3 \cdot e_1) & (e_3 \cdot e_2) & (e_3 \cdot e_3)
\end{pmatrix}
=
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]

\[ e^T e = I_{3\times3} \]
Orthonormal bases

\[ e = (e_1, e_2, e_3), \quad f = (f_1, f_2, f_3) \]

\[ f = eB \]

\[ B^T B = I_{3\times3}, \quad \det B = \pm 1 \]

Orthonormal matrix
The tensor product

Notation: \( a, b \in \mathcal{V}, \quad a \otimes b \in \text{End}(\mathcal{V}) \)

Definition: \((a \otimes b)u = a(b \cdot u)\)
The vector product

Notation: \( a, b \in \mathcal{V}, \ a \times b \in \mathcal{V} \)

\[
|a \times b| = |a||b| \sin \theta
\]

Notation: \( a \in \mathcal{V}, \ 1 \in \text{End}(\mathcal{V}), \ a \times 1 \in \text{Skew}(\mathcal{V}) \)

Definition: \( (a \times 1)u = a \times u \)
The determinant

\[
\det Q = \frac{V(Qa, Qb, Qc)}{V(a, b, c)} = \det \begin{bmatrix} Q \end{bmatrix}_e
\]

ON-basis: \( e = (e_1, e_2, e_3) \)
Chapter 4 : Rigid body kinematics

Figure 4.1 Rigid body motion.

Rigidity condition: \( |p_{AB}(t)| = \text{constant}, \forall A, B \in \mathcal{B} \)
4.1 Rigid body transplacement

\( \chi(\mathbf{r}_A) \)

Position vectors

\[ \mathbf{p}_A = \chi(\mathbf{r}_A), \ A \in \mathcal{B} \]

We choose a material point A as a reference point, called reduction point.
Rigid body transplacement

Fix point $A$ – reduction point

Transplacement: $p_A = \chi(r_A), A \in B$

Rigidity condition: $|\chi(r_B) - \chi(r_A)| = |r_B - r_A|, \forall A, B \in B$

Introduce a mapping defined by:

$$R_A(a) := \chi(r_A + a) - \chi(r_A), a \in \mathcal{V}$$

(*)

The mapping has the following properties:

$$\begin{cases} R_A(0) = 0 \\ |R_A(a) - R_A(b)| = |a - b|, \forall a, b \in \mathcal{V} \\ |R_A(a)| = |a|, \forall a \in \mathcal{V} \end{cases}$$

Isometric mapping – the length of a vector does not change!
Rigid body transplacement

Isometric mapping is linear:

Linearity: \( R_A(\alpha a + \beta b) = \alpha R_A(a) + \beta R_A(b), \quad a, b \in \mathcal{V}, \quad \alpha, \beta \in \mathbb{R} \)

Notation for linear mapping: \( R_A(a) = R_A a \) (we drop the parantheses)

Isometry implies orthonormality. \( R \) is an orthonormal transformation:

\[
R_A R_A^T = R_A^T R_A = I, \quad R_A^{-1} = R_A^T
\]

\[ (\det R_A)^2 = 1 \quad \text{two possibilities} \quad \det R_A = \pm 1 \]

The angle between the vectors does not change. Both length and angle are preserved!

\[
R_A a R_B b = a.b
\]
Rigid body transplacement

Start with: \( a = \mathbf{r}_P - \mathbf{r}_A \)

From (*) follows:

\[
\chi(\mathbf{r}_A + a) = \chi(\mathbf{r}_A) + \mathbf{R}_A a, \quad a \in \mathcal{V}
\]

\[
\chi(\mathbf{r}_P) = \chi(\mathbf{r}_A) + \mathbf{R}_A(\mathbf{r}_P - \mathbf{r}_A)
\]

\[
\mathbf{p}_P - \mathbf{p}_A = \mathbf{R}_A(\mathbf{r}_P - \mathbf{r}_A)
\]

\[
\mathbf{p}_{AP} = \mathbf{R}_A \mathbf{r}_{AP}
\]
Rigid body transplacement

\[ p_{AB} = R_A r_{AB}, \quad R_A \in SO(V) \]

Rotation tensor
What now, if we change the reduction point from $A$ to $B$?

$$b = r_B - r_A$$

Changing the reduction point

$$\chi(r_p) = \chi(r_B) + R_B(r_p - r_B)$$

What is the relationship between $R_A$ and $R_B$?

$$R_Ba = \chi(r_B + a) - \chi(r_B) = \chi(r_A + a + b) - \chi(r_A + b) = \chi(r_A + a + b) - \chi(r_A) - (\chi(r_A + b) - \chi(r_A)) = R_A(a + b) - R_Ab = R_Aa + R_Ab - R_Ab = R_Aa$$

We may thus conclude that

$$R_A = R_B = R, \quad \forall A, B \in \mathcal{B}$$

The orthogonal transformation representing a transplacement of a rigid body is independent of the reduction point!
Rigid body transplacement

\[ \chi(r_a + a) = \chi(r_a) + R_a a, \quad a \in \mathcal{V} \]

\[ \chi(r_b + a) = \chi(r_b) + R_b a, \quad a \in \mathcal{V} \]

Does the rotation tensor depend on the reduction point?

\[ R_B = R_A ? \]

No! \[ R_B = R_A = R! \]
Rigid body transplacement

Reference placement

Present placement

\[ u_A = \chi(r_A) \]

\[ p_A = \chi(r_A) - r_A \]

The displacement

\[ u(r_A) = p_A - r_A = \chi(r_A) - r_A \]
Rigid body transplacement

**Theorem 4.1** A rigid body transplacement is uniquely determined by the displacement of (an arbitrary) reduction point \( A \); \( \mathbf{u}_A \) and an orthonormal transformation \( \mathbf{R} \in SO(\mathcal{V}) \) which is independent of the reduction point.
Example 4.1 *(Plane rigid transplacement, cf. section 4.2.2)* The same rigid body transplacement is visualized in both Figure 4.6 a) and 4.6 b) below with different choices of reduction point.

Transplacement with reduction point $A$.

Transplacement with reduction point $B$.

*Figure 4.6* Plane rigid body transplacement.
Summary

Box 4.1: Rigid body transplacement

$A$: reduction point
$u_A$: displacement of the reduction point $A$
$R$: orthogonal transformation with $\det R = 1$

\[
u = u(r) = u_A + (R - I)(r - r_A)\]

\[
p = \chi(r) = \chi(r_A) + R(r - r_A)\]
The translation

A translation of a body is a transplacement where all displacements are equal, irrespective of the material point, i.e.

\[ \mathbf{u}_A = \mathbf{u}_B = \mathbf{u}_0, \quad \forall A, B \in \mathcal{B} \]

What does the associated orthonormal tensor look like?

\[ \mathbf{u}_P = \mathbf{u}_A + (\mathbf{R} - \mathbf{I})r_{AP} \]

\[ (\mathbf{R} - \mathbf{I})a = 0 \]

\[ \mathbf{R} = \mathbf{I} \]
The translation

**Proposition 4.2** A translation is a rigid transplacement determined by a displacement \( u_0 \), independent of the reduction point, and an orthonormal transformation \( R \) equal to the unit tensor.

\[
\mathbf{u}_A = \mathbf{u}_B = \mathbf{u}_0, \quad \forall A, B \in \mathcal{B},
\]

![Diagram of translation](image)

Figure 4.7 The translation.

**Box 4.2: Translation**

- \( A \): *reduction point*
- \( \mathbf{u}_A = \mathbf{u}_0 \): *displacement of the reduction point A*
- \( R = 1 \): *orthonormal transformation*
- \( \mathbf{u} = \mathbf{u}_0 \): *displacement of the body*
The rotation

Consider a rotation of the body around a fixed axis in space through the point

Rotation axis: \((A, n)\)

introduce a Right-handed Ortho-Normal basis (a ‘RON-basis’)

\[ i = (i_1, i_2, i_3) \]

Decomposition:

\[ r_{AB} = r_{AB}^\perp + n(n \cdot r_{AB}) \]

Then we write

\[ r_{AB}^\perp = i_1 x_1 + i_2 x_2 \]
The rotation

Figure 4.8 Rotation around a fixed axis.

Figure 4.9 Rotation around a fixed axis.

\[ r_{AB} = r_{AB}^\perp + n(n \cdot r_{AB}) \]

\[ r_{AB}^\perp = i_1 x_1 + i_2 x_2 \]

The vector corresponding to \( r_{AB} \) in the present placement is:

\[ p_{AB} = p_{AB}^\perp + n(n \cdot p_{AB}) \]

\[ p_{AB}^\perp = e_1 x_1 + e_2 x_2 \]

\[ |r_{AB}| = |p_{AB}| = x_1^2 + x_2^2 + (n \cdot r_{AB})^2 \]
The rotation

\[ e = (e_1, e_2, e_3), \quad e_3 = n \]

is a RON-basis. This basis is related to \( i \) by

\[
\begin{align*}
e_1 &= i_1 \cos \varphi + i_2 \sin \varphi \\
e_2 &= i_1 (-\sin \varphi) + i_2 \cos \varphi, \quad 0 \leq \varphi < 2\pi \\
e_3 &= i_3 = n
\end{align*}
\]

This can be expressed in matrix format.

**Figure 4.9** Rotation around a fixed axis.
The rotation matrix

\[(e_1, e_2, e_3) = (i_1, i_2, i_3) \left( \begin{array}{ccc}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{array} \right), \quad e = i[R]_i\]

\[\left[ R \right]_i = \left( \begin{array}{ccc}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{array} \right)\]

The matrix representation of the rotation \( R \) is called the canonical representation.

\[\left[ R \right]_i^T \left[ R \right]_i = \left[ R \right]_i \left[ R \right]_i^T = [I], \quad \det \left[ R \right]_i = 1\]
The rotation tensor

Now let \( \mathbf{R} \) be the tensor defined by:

\[
\begin{align*}
R\mathbf{i}_1 &= \mathbf{e}_1 \\
R\mathbf{i}_2 &= \mathbf{e}_2 \\
R\mathbf{i}_3 &= \mathbf{e}_3 = \mathbf{i}_3 = n
\end{align*}
\]

\[
\mathbf{p}_{AB} = \mathbf{p}_{AB}^\perp + n(n \cdot \mathbf{r}_{AB}) = R\mathbf{i}_1 x_1 + R\mathbf{i}_2 x_2 + Rn(n \cdot \mathbf{r}_{AB}) = R(i_1 x_1 + i_2 x_2 + n(n \cdot \mathbf{r}_{AB})) = R\mathbf{r}_{AB}
\]

\[
\mathbf{p}_{AB} = R\mathbf{r}_{AB}
\]

which means that the rotation is a rigid transplacement.
Box 4.3: Rotation

$(A, n)$: rotation axis

$A$: reduction point

$u_A = 0$: displacement of the reduction point $A$

$R$: orthonormal transformation, $[R] = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\phi$: rotation angle $0 \leq \phi < 2\pi$

$i = (i_1, i_2, i_3)$: referential base, $i_3 = n = e_3$
Theorem 4.2 (The Euler theorem, 1775) Every rigid transplacement with a fixed point is equal to a rotation around an axis through the fixed point.
The canonical representation

\[ \mathbf{i} = (i_1, i_2, i_3), \quad i_3 = \mathbf{n} \]

axis \((A, \mathbf{n})\)

\[ \mathbf{n} \]

\[ \mathbf{i} \]

\[ \mathbf{n} \]

\[ \mathbf{B}_0 \]

\[ \mathbf{i} \text{ is vector perpendicular to } \mathbf{n} \]

The transplacement is given by:

\[ p_{AB} = R r_{AB}, \quad \forall B \in \mathcal{B} \]

We now try to find out if there are any points \(C\) with the property:

\[ p_{AC} = R r_{AC} = r_{AC} \]

If this is true then there are material vectors unaffected by the transplacement. This brings us to the eigenvalue problem for the tensor \(R\)

\[ R \mathbf{n} = \lambda \mathbf{n} \]
The characteristic equation, eigenvalues

\[ p_R(\lambda) = \det(\lambda I - R) = \det(\lambda [I] - [R]) = \]

\[
\begin{bmatrix}
\lambda - \cos \varphi & \sin \varphi & 0 \\
-\sin \varphi & \lambda - \cos \varphi & 0 \\
0 & 0 & \lambda - 1
\end{bmatrix}
\]

\[= (\lambda - 1)((\lambda - \cos \varphi)^2 + \sin^2 \varphi)\]

\[ p_R(\lambda) = 0 \iff \lambda_1 = 1, \ \lambda_{2,3} = e^{\pm i\varphi} \]
The canonical representation

\( \mathbf{i} = (i_1, i_2, i_3), \ i_3 = n \)

Eigen value problem: find a material points \( C \) such that \( r_{AC} \) is an eigenvector to \( R \) with the corresponding eigenvalue 1.

\( Rn = \lambda n \)

\[ \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} \]
The canonical representation

\[
(i_1 \cos \varphi + i_2 \sin \varphi)(i_1 (-\sin \varphi) + i_2 \cos \varphi) =
\]

\[
(i_1 \cos \varphi + i_2 \sin \varphi) \otimes i_1 + (i_1 (-\sin \varphi) + i_2 \cos \varphi) \otimes i_2 + i_3 \otimes i_3 =
\]

Equivalent to the following tensor-product representation of \( R \)

\[
R = i_3 \otimes i_3 + (i_1 \otimes i_1 + i_2 \otimes i_2) \cos \varphi + (i_2 \otimes i_1 - i_1 \otimes i_2) \sin \varphi
\]

\[
n \otimes n + \cos \varphi(1 - n \otimes n) + \sin \varphi n \times 1
\]
Box 4.5: Rigid transplacement

\[ p = \chi(r) = r_A + u_A + R(r - r_A) \]

- \( A \): reduction point
- \( r_A \): position vector of the reduction point in the reference placement
- \( u_A \): displacement of the reduction point
- \( r \): position vector of a material point in the reference placement
- \( p \): position vector of a material point in the present placement
- \( R \): rotation tensor
Screw representation of the transplacement

Since the translation $\mathbf{u}_A$ changes with the reduction point A and the rotation $\mathbf{R}$ is independent of this choice we may ask if there is a specific reduction point S such that the displacement $\mathbf{u}_S$ is parallel to the rotation axis.

\[
\mathbf{p} = \mathbf{r}_S + n\mathbf{s} + \mathbf{R}(\mathbf{r} - \mathbf{r}_S)
\]

Screw line
\[
\mathcal{L}_S = \left\{ S \in \mathcal{B} \left| \mathbf{r}_S = \mathbf{r}_A + \mathbf{r}_{AS}^\perp + n\sigma, \quad \sigma \in \mathcal{I} \right. \right\}
\]

\[
\mathbf{r}_{AS}^\perp = -\frac{(\mathbf{R} - I)^T \mathbf{u}_A}{2(1 - \cos \varphi)}, \quad \varphi \neq 0
\]

**Theorem 4.3** (Chasles’ Theorem, 1830) Every transplacement of a rigid body can be realized by a rotation about an axis combined with a translation of minimum magnitude parallel to that axis. The reduction point for this so-called screw transplacement is located on the screw line given in (4.28).
The special orthogonal group

The set of all orthonormal tensors on \( \mathcal{V} \) forms an algebraic group, the orthonormal group,

\[
O(\mathcal{V}) = \left\{ \mathbf{Q} \left| \mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = \mathbf{I} \right. \right\}
\]

Being a group means that

\[
\begin{aligned}
\mathbf{Q}_1, \mathbf{Q}_2 \in O(\mathcal{V}) & \Rightarrow \mathbf{Q}_2 \mathbf{Q}_1 \in O(\mathcal{V}) \\
\mathbf{I} \in O(\mathcal{V}) \\
\mathbf{Q}^{-1} = \mathbf{Q}^T \in O(\mathcal{V})
\end{aligned}
\]

However, in general we have \( \mathbf{Q}_1 \mathbf{Q}_2 \neq \mathbf{Q}_2 \mathbf{Q}_1 \) and we say that the operation of combining orthonormal tensors is non-commutative.

The set of all rotations forms a subgroup in \( O(\mathcal{V}) \) called the special orthonormal group

\[
SO(\mathcal{V}) = \left\{ \mathbf{R} \in O(\mathcal{V}) \left| \det \mathbf{R} = 1 \right. \right\}
\]
Non-commutativity of rotations

Combining rotations is thus a non-commutative operation as can be seen from the figure below. The special orthonormal group has the mathematical structure of a so-called Lie-group.
Chapter 4.3 The main representation theorem

Theorem 4.1  To every rotation tensor \( R \in SO(V) \) there exists a unit vector \( n \) and an angle \( \varphi \in [0, 2\pi) \) so that

\[
R = R(n, \varphi) = n \otimes n + \cos \varphi (I - n \otimes n) + \sin \varphi n \times 1
\]

This representation is mainly unique. If \( R = I \) then \( n \) may be arbitrarily chosen, while \( \varphi = 0 \) (modulo \( 2\pi \)). If \( R \neq I \) then \( n \) is uniquely determined, apart from a factor \( \pm I \). For a given \( n \) the angle \( \varphi \) is uniquely determined (modulo \( 2\pi \)). For a definition of the tensor \( a \times 1 \in \text{End}(V) \) see Appendix A.4.39.
The main representation theorem using components

\[ R = R(n, \varphi) = n \otimes n + \cos \varphi (1 - n \otimes n) + \sin \varphi n \times 1 \]

\[ \underline{e} = (e_1 \ e_2 \ e_3) \quad n = \underline{e} [n]_e \quad [n]_e = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \]

\[ [n \otimes n]_e = [n]_e [n]_e^T = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} = \begin{pmatrix} n_1 n_1 & n_1 n_2 & n_1 n_3 \\ n_2 n_1 & n_2 n_2 & n_2 n_3 \\ n_3 n_1 & n_3 n_2 & n_3 n_3 \end{pmatrix} \]

\[ [n \times 1]_e = [n]_e \times [1]_e = \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \]
The main representation theorem using components

\[ R = n \otimes n + \cos \varphi (1 - n \otimes n) + \sin \varphi n \times 1 \]

\[ e = (e_1, e_2, e_3), \quad n = e[n]_e, \quad [n]_e = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \]

\[ [R]_e = [n \otimes n]_e + \cos \varphi [(1 - n \otimes n)]_e + \sin \varphi [n \times 1]_e = \]

\[ [n]_e [n]_e^T + \cos \varphi ([1]_e - [n]_e [n]_e^T) + \sin \varphi [n \times 1]_e \]
How to calculate the rotation axis $\mathbf{n}$ and the rotation angle $\varphi$ from the rotation tensor $\mathbf{R}$

$$
\mathbf{R} = \mathbf{n} \otimes \mathbf{n} + \cos \varphi (\mathbf{1} - \mathbf{n} \otimes \mathbf{n}) + \sin \varphi \mathbf{n} \times \mathbf{1}
$$

$\mathbf{R} \rightarrow \mathbf{n}, \varphi$?

$$
\mathbf{b} = (b_1, b_2, b_3) \quad \text{arbitrary RON-basis}
$$

Eigenvalue problem

$$
\mathbf{R} \mathbf{n} = \mathbf{n} \iff (\mathbf{R} - \mathbf{1}) \mathbf{n} = \mathbf{0} \iff (\left[ \mathbf{R} \right]_\mathbf{b} - [\mathbf{1}]) \left[ \mathbf{n} \right]_\mathbf{b} = [\mathbf{0}]
$$

$$
\left[ \mathbf{R} \right]_\mathbf{b} = \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix} \in \mathbb{R}^{3 \times 3}
$$

$$
\left[ \mathbf{n} \right]_\mathbf{b} = \begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix} \in \mathbb{R}^{3 \times 1}
$$
The eigenvalue problem

\[
\begin{pmatrix}
R_{11} - \lambda & R_{12} & R_{13} \\
R_{21} & R_{22} - \lambda & R_{23} \\
R_{31} & R_{32} & R_{33} - \lambda
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

\[
p_R(\lambda) = \det \begin{pmatrix}
R_{11} - \lambda & R_{12} & R_{13} \\
R_{21} & R_{22} - \lambda & R_{23} \\
R_{31} & R_{32} & R_{33} - \lambda
\end{pmatrix}
\]

\[
p_R(\lambda) = 0 \iff \lambda_1 = 1, \; \lambda_{2,3} = e^{\pm i \phi}
\]

\[
\cos \varphi = \frac{\lambda_2 + \lambda_3}{2}, \quad \sin \varphi = \frac{\lambda_2 - \lambda_3}{2i}
\]
The eigenvalue problem

\[ \lambda_1 = 1 \]

\[
\begin{pmatrix}
R_{11} - 1 & R_{12} & R_{13} \\
R_{21} & R_{22} - 1 & R_{23} \\
R_{31} & R_{32} & R_{33} - 1
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix}
\Rightarrow n
\]

\[
\begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[ trR = R_{11} + R_{22} + R_{33} = 1 + 2 \cos \varphi \]

\[
\varphi = \begin{cases} 
\arccos \left( \frac{\text{tr}R - 1}{2} \right) = \arccos \left( \frac{R_{11} + R_{22} + R_{33} - 1}{2} \right) \\
2\pi - \arccos \left( \frac{R_{11} + R_{22} + R_{33} - 1}{2} \right)
\end{cases}
\]
Coordinate representations of $SO(V)$

- Euler angles
- Bryant angles
- Euler parameters
- Quaternions
- ......
Euler angles: \((\psi, \theta, \phi)\)

\[ \underline{e} = R \underline{e}^o = \underline{e}^o [R]_{e^o} \]
Factorization: \((\psi, \theta, \phi)\)

\[
R = R_3(\phi) R_1(\theta) R_3(\psi)
\]

Every rotation may be written as a product of three simple rotations.
Euler angles: First rotation $\psi$

We start the factorization by introducing the first rotation $R_3(\psi) : e_3^0 \psi$ which brings the base $e^o$ to the base $f = (f_1, f_2, f_3)$ by rotating $e^o$ around the axis with direction $e_3^o$ angle $\psi$.

$$
\begin{align*}
\vec{f} &= R_3(\psi) \vec{e}^o = \vec{e}^o \left[ R_3(\psi) \right]_{\vec{e}^o}, & \left[ R_3(\psi) \right]_{\vec{e}^o} &= \begin{pmatrix}
\cos \psi & -\sin \psi & 0 \\
\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}, \\
\vec{e}^o &= (e_1^o, e_2^o, e_3^o) \\
\vec{f} &= (f_1, f_2, f_3)
\end{align*}
$$
Euler angles: Second rotation $\theta$

Next we take the rotation $R_1(\theta) : f_1 \theta$ which brings $\underline{f}$ to the RON-basis $\underline{g} = (g_1, g_2, g_3)$ by a rotation with the angle $\theta$ around an axis with the direction $f_1$, the so-called intermediate axis.

$$\underline{g} = R_1(\theta) \underline{f} = \underline{f} \begin{bmatrix} R_1(\theta) \end{bmatrix}_{\underline{f}} ,$$

$$\begin{bmatrix} R_1(\theta) \end{bmatrix}_{\underline{f}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$
Euler angles: Third rotation $\phi$

Finally we take the rotation $R_3(\phi) : g_3 \phi$ which brings $g$ to the final basis $e$ by a rotation the angle $\phi$ around an axis with direction $g_3$.

\[ e = R_3(\phi)g = g[R_3(\phi)]_g, \quad [R_3(\phi)]_g = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
Euler angles: All rotations in combination

\[
\begin{align*}
\bar{f} &= R_3(\psi) \bar{e}^o = e^o \left[ R_3(\psi) \right]_{e^o} \\
\bar{g} &= R_1(\theta) \bar{f} = f \left[ R_1(\theta) \right]_{f} \\
\bar{e} &= R_3(\phi) \bar{g} = g \left[ R_3(\phi) \right]_{g}
\end{align*}
\]

\[\downarrow\]

Re-substitution yields:

\[
\bar{e} = R \bar{e}^o = g \left[ R_3(\phi) \right]_{g} = f \left[ R_1(\theta) \right]_{f} \left[ R_3(\phi) \right]_{g} = e^o \left[ R_3(\psi) \right]_{e^o} \left[ R_1(\theta) \right]_{f} \left[ R_3(\phi) \right]_{g}
\]

\[
\begin{align*}
\bar{e} &= R \bar{e}^o = e^o \left[ R_3(\psi) \right]_{e^o} \left[ R_1(\theta) \right]_{f} \left[ R_3(\phi) \right]_{g}
\end{align*}
\]
Euler angles: The rotation

\[ \mathbf{e}_o^g \rightarrow \mathbf{f} \rightarrow \mathbf{g} \rightarrow \mathbf{e} \]

\[
[R]_{\mathbf{e}_o^g} = [R_3(\psi)]_{\mathbf{e}_o^g}[R_1(\theta)]_{\mathbf{f}}[R_3(\phi)]_{\mathbf{g}} =
\]

\[
\begin{pmatrix}
\cos\psi & -\sin\psi & 0 \\
\sin\psi & \cos\psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos\theta & -\sin\theta \\
0 & \sin\theta & \cos\theta
\end{pmatrix}
\begin{pmatrix}
\cos\phi & -\sin\phi & 0 \\
\sin\phi & \cos\phi & 0 \\
0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
\cos\psi \cos\phi - \sin\psi \cos\theta \sin\phi & -\cos\psi \sin\phi - \sin\psi \cos\theta \cos\phi & \sin\psi \sin\theta \\
\sin\psi \cos\phi + \cos\psi \cos\theta \sin\phi & -\sin\psi \sin\phi + \cos\psi \cos\theta \cos\phi & -\cos\psi \sin\theta \\
\sin\theta \sin\phi & -\sin\theta \sin\phi & \sin\theta \cos\phi & \cos\theta
\end{pmatrix}
\]
Theorem 4.2 A rotation $R$ is determined by a set of three real coordinates, called the Euler angles: $(\psi, \theta, \phi) \in \mathbb{R}^3$, $0 \leq \psi < 2\pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. Conversely these angles are essentially determined by $R$. In fact the angle $\theta \in [0, \pi]$ is uniquely determined by $R$. If $\theta \in ]0, \pi[$ then also $\psi, \phi \in [0, 2\pi[$ are uniquely determined by $R$. When $\theta = 0$ only $\psi + \phi$ is uniquely determined and when $\theta = \pi$ only $\psi - \phi$. 
Euler angles: Singularity

Special case: \( \theta = 0, \pi \)

\[
[R]_{\theta} = \begin{pmatrix}
\cos \psi \cos \phi \pm \sin \psi \sin \phi & -\cos \psi \sin \phi \pm \sin \psi \cos \phi & 0 \\
\sin \psi \cos \phi \pm \cos \psi \sin \phi & -\sin \psi \sin \phi \pm \cos \psi \cos \phi & 0 \\
0 & 0 & \pm 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cos(\psi \pm \phi) & -\sin(\psi \pm \phi) & 0 \\
\sin(\psi \pm \phi) & -\cos(\psi \pm \phi) & 0 \\
0 & 0 & \pm 1
\end{pmatrix}
= \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\]

Only \( \psi \pm \phi \) is determined. Singular!
Euler angles: Singularity

\[
[R]_e^o = \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\cos \psi \cos \phi - \sin \psi \cos \theta \sin \phi & -\cos \psi \sin \phi - \sin \psi \cos \theta \cos \phi & \sin \psi \sin \theta \\
\sin \psi \cos \phi + \cos \psi \cos \theta \sin \phi & -\sin \psi \sin \phi + \cos \psi \cos \theta \cos \phi & -\cos \psi \sin \theta \\
\sin \theta \sin \phi & \sin \theta \cos \phi & \cos \theta
\end{pmatrix}
\]

\[
\begin{aligned}
\cos \theta &= R_{33}, & \sin \theta &= \sqrt{1 - \cos^2 \theta} \\
\cos \psi &= -\frac{R_{23}}{\sin \theta}, & \sin \psi &= \frac{R_{13}}{\sin \theta} \\
\cos \phi &= \frac{R_{32}}{\sin \theta}, & \sin \phi &= \frac{R_{31}}{\sin \theta}
\end{aligned}
\]

\[
\theta = 0, \pi \quad \text{singular}
\]

These formulas show that numerical difficulties are to be expected for values of \( \theta \) close to \( n\pi, \ n = 0, 1, .. \).
The rotation angle

\[ trR = 1 + 2 \cos \varphi = (1 + \cos \theta) \cos(\phi + \psi) + \cos \theta \]

\[
\cos \varphi = \frac{(1 + \cos \theta) \cos(\phi + \psi) + \cos \theta - 1}{2}
\]

**Figure 4.5** The gyroscope.
Chapter 4.3.3 Euler parameters

Start from the fundamental representation:

\[ R = n \otimes n + \cos \varphi (1 - n \otimes n) + \sin \varphi n \times 1 = \]

\[ 1 + (1 - \cos \varphi)(n \otimes n - 1) + \sin \varphi n \times 1 \]

use of the trigonometric identities

\[ \sin^2 \frac{\varphi}{2} = \frac{1 - \cos \varphi}{2} \quad \sin \varphi = 2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \]

\[ R = 1 + 2 \sin^2 \frac{\varphi}{2} n \times (n \times 1) + 2 \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} n \times 1 \]

Introduce:

\[ e_0 = \cos \frac{\varphi}{2}, \quad e = n \sin \frac{\varphi}{2} \]
Euler parameters

then we have

$$R = 1 + 2(e \times 1)(e \times 1) + 2e_0e \times 1$$

With a specific RON-basis $b = (b_1 \ b_2 \ b_3)$, RON-basis,

in which

$$n = b_1n_1 + b_2n_2 + b_3n_3, \quad |n| = 1 \quad e_0 = \cos \frac{\varphi}{2}, \quad e = n \sin \frac{\varphi}{2}$$

Then we have the following relations:

$$\begin{cases}
  e = b_1e_1 + b_2e_2 + b_3e_3 \\
  e_0 = \cos \frac{\varphi}{2}, \quad e_1 = n_1 \sin \frac{\varphi}{2}, \quad e_2 = n_2 \sin \frac{\varphi}{2}, \quad e_3 = n_3 \sin \frac{\varphi}{2}
\end{cases}$$

$$n.n = 1 \quad \Rightarrow \quad \begin{cases}
  e = (e_0, e_1, e_2, e_3) \in \mathbb{R}^4, \quad e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \\
  S^3 = \{ e \in \mathbb{R}^4 | e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \}
\end{cases}$$

This constraint on the Euler parameters represents the unit sphere

The parameters $e = (e_0, e_1, e_2, e_3)$ are known as the Euler parameters.
How to calculate matrix elements from Euler parameters

To indicate the Euler parameters associated with a rotation we write

\[
R = R(e) = R(e_0, e) = I + 2(e \times I)^2 + 2e_0 e \times I, \quad e = (e_0, e) \in S^3
\]

\[
_b = (b_1 \ b_2 \ b_3), \ \text{RON-basis,}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
+ 2
\begin{pmatrix}
0 & -e_3 & e_2 \\
e_3 & 0 & -e_1 \\
-e_2 & e_1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & -e_3 & e_2 \\
e_3 & 0 & -e_1 \\
-e_2 & e_1 & 0
\end{pmatrix}
\]

\[
2e_0
\begin{pmatrix}
0 & -e_3 & e_2 \\
e_3 & 0 & -e_1 \\
-e_2 & e_1 & 0
\end{pmatrix} =
\begin{pmatrix}
2(e_0^2 + e_1^2) - 1 & 2(e_1e_2 - e_0e_3) & 2(e_1e_3 + e_0e_2) \\
2(e_1e_2 + e_0e_3) & 2(e_0^2 + e_2^2) - 1 & 2(e_2e_3 - e_0e_1) \\
2(e_1e_3 - e_0e_2) & 2(e_2e_3 + e_0e_1) & 2(e_0^2 + e_3^2) - 1
\end{pmatrix}
\]
How to calculate Euler parameters from matrix elements

\[
[R]_{e^o} = \begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\]

\[
\begin{cases}
e_0^2 = \frac{\text{tr}R + 1}{4} = \frac{R_{11} + R_{22} + R_{33} + 1}{4} \\
e_i^2 = \frac{R_{ii} - \text{tr}R - 1}{2} = \frac{1 + 2R_{ii} - R_{11} - R_{22} - R_{33}}{4}, \quad i = 1, 2, 3
\end{cases}
\]

No singularity! (In contrast to the Euler angles)

There is a 2-1 correspondence between Euler parameters and rotation matrices!
How to calculate Euler parameters from matrix elements, continued

\[ e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \Rightarrow (e_0, e_1, e_2, e_3) \neq (0, 0, 0, 0) \]

\[ e_0 = \pm \sqrt{\frac{R_{11} + R_{22} + R_{33} + 1}{4}} \neq 0 \]

\[ e_1 = \frac{R_{32} - R_{23}}{4e_0}, \quad e_2 = \frac{R_{13} - R_{31}}{4e_0}, \quad e_3 = \frac{R_{21} - R_{12}}{4e_0} \]
How to calculate Euler parameters from matrix elements

\[ e_0 = \pm \sqrt{\frac{R_{11} + R_{22} + R_{33} + I}{4}} = 0 \]

\[ 4e_1e_2 = R_{21} + R_{12}, \quad 4e_1e_3 = R_{31} + R_{13}, \quad 4e_2e_3 = R_{32} + R_{23} \]

\[ e_0 = 0, \quad e_1 = \pm \sqrt{\frac{I + R_{11} - R_{22} - R_{33}}{4}} \neq 0 \]

\[ e_2 = \frac{R_{21} + R_{12}}{4e_1}, \quad e_3 = \frac{R_{31} + R_{13}}{4e_1} \]
How to calculate Euler parameters from matrix elements

Special cases:

\[ e_0 = 0, \; e_1 = 0, \; e_2 = \pm \sqrt{\frac{1 + R_{22} - R_{11} - R_{33}}{4}} \neq 0 \]

\[ e_3 = \frac{R_{31} + R_{13}}{4e_2} \]

\[ e_0 = 0, \; e_1 = 0, \; e_2 = 0, \; e_3 = \pm \sqrt{\frac{1 + R_{33} - R_{11} - R_{22}}{4}} \neq 0 \]
Chapter 4.3.4 The composition of rotations

Let \( R_1 \) and \( R_2 \) be two rotations. The composition of \( R_1 \) and \( R_2 \) is given by

\[
R = R_2 R_1
\]

and this represents another rotation. We have the following

**Proposition 4.1** If \( R_1 = R_1(e_{0,1}, e_1) \), \( R_2 = R_2(e_{0,2}, e_2) \) and \( R = R_2 R_1 \) then \( R = R(e_0, e) \) where

\[
e_0 = e_{0,1}e_{0,2} - e_1 \cdot e_2, \quad e = e_2 e_{0,1} + e_1 e_{0,2} + e_1 \times e_2
\]  \hspace{1cm} (4.18)

**Corollary 4.1** Two rotations commute if and only if their rotation axes are parallel, i.e.

\[
R_2 R_1 = R_1 R_2 \iff e_1 \parallel e_2
\]
Chapter 4.3.5 The exponential map

Rotation \( R : n\varphi \in SO(\mathcal{V}) \)

Skew-sym. Tensor \( N = n \times 1 \in Skew(\mathcal{V}) \)

Expanding in Maclaurin series:

\[
R = I + \varphi N + \frac{\varphi^2}{2!} N^2 + \frac{\varphi^3}{3!} N^3 + ... = \exp(\varphi N) = \exp(n\varphi \times I)
\]

The exponential map:

\[
\exp(A) := I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + ..., \quad A \in \text{End}(\mathcal{V})
\]
Chapter 4.4 Time-dependent transplacement, Motion – velocity and acceleration

The rigid motion is a set of rigid transplacements parameterized by time $t$.
Rigid body velocity

Spin tensor: \[ W = \dot{R} R^T \]
Angular velocity: \[ \omega = \alpha x (\dot{R} R^T) \]
(axial vector of \( W \))

anti-symmetric tensor, since:

\[ W^T = (\dot{R} R^T)^T = R \dot{R}^T = -\dot{R} R^T = -W \]

\[ R R^T = I \Rightarrow \dot{R} R^T + R \dot{R}^T = 0 \Rightarrow R \dot{R}^T = -\dot{R} R^T \]

Rigid body velocity field: \[ \mathbf{v}_P = \mathbf{v}_A + \omega \times \mathbf{p}_{AP}, \ P \in \mathcal{B} \]
Rigid body rotation

Spin tensor: \( W = \dot{R}R^T \)

Angular velocity: \( \omega = \alpha x (\dot{R}R^T) \)

\[
W = \dot{R}R^T \iff \dot{R} = WR = (\omega \times I)R = \omega \times R
\]

\[
\begin{cases}
\dot{R} = \omega \times R \\
R(0) = R_0
\end{cases} \iff R = R(t) = (\exp(\int^t_0 \omega(s)ds \times I))R_0
\]

Thus, if the angular velocity \( \omega = \omega(t) \) is known and the rotation tensor is known at time \( t = 0 \) then the rotation tensor : \( R = R(t) \) is known.
Rigid body acceleration

\[ a_P = a_A + \alpha \times p_{AP} + \omega \times (\omega \times p_{AP}), \ P \in \mathcal{B} \]

It is more difficult to graphically illustrate the acceleration field than the velocity field. We look at the special case when the body rotates around an axis with fixed direction

\[ \omega = n \omega, \ \alpha = n \dot{\omega}, \ \dot{n} = 0 \]

The tangential and normal parts of the relative acceleration:

\[ |\alpha \times p_{AP}| = |\dot{\omega}| |n \times p_{AP}| = |\dot{\omega}| |p_{AP}| \sin \beta = |\dot{\omega}| d \]
\[ |\omega \times (\omega \times p_{AP})| = |\omega| |\omega \times p_{AP}| = |\omega|^2 |p_{AP}| \sin \beta = \omega^2 d \]
4.4.2 Euler-Poisson velocity formula for moving bases

Starting from a fixed RON-base
In the reference placement

\[ e^0 = (e_1^0, e_2^0, e_3^0) \]

RON-basis following the rotation of the rigid body

\[ e_1 = Re_1^0, \ e_2 = Re_2^0, \ e_3 = Re_3^0 \]

\[ [R]_{e_0} = [R]_e \]
Euler-Poisson velocity formula for moving bases

**Theorem 4.4** (Euler-Poisson velocity formula for moving bases)

\[ \dot{e}_i = \omega \times e_i, \quad i = 1, 2, 3 \]  \hspace{1cm} (4.32)

where

\[ \omega = \frac{1}{2} \sum_{i=1}^{3} e_i \times \dot{e}_i \]  \hspace{1cm} (4.33)
Euler-Poisson velocity formula for moving vectors and tensors

\[ a = a(t) = \sum_{i=1}^{3} e_i(t) a_i(t) \quad \left( \dot{a} \right)_e = \sum_{i=1}^{3} e_i \dot{a}_i = \dot{a}^{rel} \]

\[ \dot{a} = \omega \times a + \dot{a}^{rel} \]

\[ A = A(t) = \sum_{i,j=1}^{3} e_i(t) \otimes e_j(t) A_{ij}(t) \quad \left( \dot{A} \right)_e = \sum_{i,j=1}^{3} e_i(t) \otimes e_j(t) \dot{A}_{ij}(t) = \dot{A}^{rel} \]

\[ \dot{A} = \omega \times A - A(\omega \times I) + \dot{A}^{rel} \]
Angular acceleration

moving vector

\[ \ddot{a} = \omega \times a + \ddot{a}^{rel} \]

\[ \omega = \sum_{i=1}^{3} e_i \omega_i \]

\[ \dot{\omega} = \omega \times \omega + \dot{\omega}^{rel} = \dot{\omega}^{rel} \]

\[ \dot{\omega} = \dot{\omega}^{rel} = \sum_{i=1}^{3} e_i \dot{\omega}_i \]
Angular velocity and Euler angles

**Proposition 4.6** The angular velocity \( \omega \), expressed in Euler angles, is given by

\[
\omega = e_1 \omega_1 + e_2 \omega_2 + e_3 \omega_3
\]

where

\[
\begin{align*}
\omega_1 &= \dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi \\
\omega_2 &= \dot{\psi} \sin \theta \cos \phi - \dot{\theta} \sin \phi \\
\omega_3 &= \dot{\psi} \cos \theta + \dot{\phi}
\end{align*}
\]  

and \( e = (e_1, e_2, e_3) \) is an RON-basis following the rotation of the rigid body (‘fixed to the body’).
Combining rotations
The addition of angular velocities, version 1

consider two rotation tensors

\[ R_1 = R_1(t) \quad R_2 = R_2(t) \]

the combined rotation tensor

\[ R = R_2 R_1 \]

We introduce the angular velocities

\[ \omega_1 = ax(\dot{R}_1 R_1^T) \quad \omega_2 = ax(\dot{R}_2 R_2^T) \quad \omega = ax(\dot{R} R^T) \]

\[ \omega = \omega_2 + R_2 \omega_1 \]

Addition rule 1 – not used that often
Combining rotations
The addition of angular velocities, version 2

\[ f = R_1 e_0 \quad g = R_2 f \]

\[ g = Re_0 \quad R = R_2 R_1 \]

\[ \omega_f = \omega_{fe_0} = ax(\dot{R}_1 R_1^T) \quad \omega_{gf} = ax((\dot{R}_2)_f R_2^T) \quad \omega = \omega_{ge_0} = ax(\dot{R} R^T) \]
Combining rotations
The addition of angular velocities, version 2

\[ \omega_{ge_0} = \omega_{gf} + \omega_{fe_0} \]

Addition rule 2
Box 4.8: Velocity and acceleration

\[
\begin{align*}
    \mathbf{v} &= \mathbf{v}_A + \mathbf{\omega} \times (\mathbf{p} - \mathbf{p}_A) \\
    \mathbf{a} &= \mathbf{a}_A + \mathbf{\alpha} \times \mathbf{p} + \mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{p}) \\
    \mathbf{W} &= \dot{\mathbf{R}} \mathbf{R}^T, \quad \mathbf{\omega} = ax(\mathbf{W}) \\
    \mathbf{\alpha} &= \dot{\mathbf{\omega}} \\
    \dot{\mathbf{e}}_i &= \mathbf{\omega} \times \mathbf{e}_i, \quad i = 1, 2, 3 \\
    \dot{\mathbf{u}} &= \mathbf{\omega} \times \mathbf{u} + \dot{\mathbf{u}}^{rel} \\
    \dot{\mathbf{A}} &= \mathbf{\omega} \times \mathbf{A} - \mathbf{A} (\mathbf{\omega} \times \mathbf{I}) + \left( \mathbf{\dot{A}} \right)_E \\
\end{align*}
\]

\[
\begin{align*}
    \mathbf{\omega} &= e_1 \omega_1 + e_2 \omega_2 + e_3 \omega_3 \\
    \omega_1 &= \dot{\psi} \sin \theta \sin \phi + \dot{\theta} \cos \phi \\
    \omega_2 &= \dot{\psi} \sin \theta \cos \phi - \dot{\theta} \sin \phi \\
    \omega_3 &= \dot{\psi} \cos \theta + \dot{\phi} \\
    \mathbf{e} &= (e_1 \ e_2 \ e_3) \\
    \mathbf{\omega} &= \mathbf{\omega}_2 + \mathbf{R}_2 \mathbf{\omega}_1 \\
    \mathbf{\omega}_{g_{e^0}} &= \mathbf{\omega}_{g_{f}} + \mathbf{\omega}_{f_{e^0}}
\end{align*}
\]

- Rigid body velocity field
- Rigid body acceleration field
- Spin tensor, Angular velocity
- Angular acceleration
- Euler – Poisson for vectors
- Absolute and relative velocity
- Euler – Poisson for tensors
- Angular velocity in Euler angles
- RON - basis fixed to the body
- Addition of angular velocities (1)
- Addition of angular velocities (2)
Example 4.2 Consider a homogeneous, rigid wheel, with radius \( r \), rolling on a plane surface with centre of mass velocity \( \mathbf{v}_c = i v \) and acceleration \( \mathbf{a}_c = i a \) \( (a = \dot{v}) \) directed along the plane. The basis \( \mathbf{i} = (i \ j \ k) \) is such that \( i \) is horizontal and parallel to the wheel, \( j \) is vertical and \( k \) is perpendicular to the wheel, see Figure 4.10. The angular velocity of the basis \( \mathbf{i} \) is given by \( \omega_i = j \omega_p \). The angular velocity of the wheel is given by \( \omega = \omega_i + k \omega_r = j \omega_p + k \omega_r \). Now let \( C \) be the material point in the wheel, momentarily in contact with the surface.

![Diagram of a rolling wheel with labels for \( \omega_i \), \( \omega_p \), \( \omega_r \), \( v_c \), and \( a_c \).](image)

Figure 4.10 The rolling wheel.
Euler angles, first rotation

\[ \mathbf{f} = \mathbf{R}_3(\psi) \mathbf{e}^o = \mathbf{e}^o \left[ \mathbf{R}_3(\psi) \right]_{e^o}, \]

\[ \omega_1 = \omega_{f e^o} = ax(\mathbf{R}_1 \mathbf{R}_1^T) \]

\[ \left[ \mathbf{R}_3(\psi) \right]_{e^o} = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

\[ \mathbf{R} = \mathbf{R}_3(\phi)\mathbf{R}_1(\theta)\mathbf{R}_3(\psi) \]

**Example 4.2 (Euler angles revisited)** Considering Euler angles we have, according to (4.6),

\[ \omega_{f e^o} = ax(\mathbf{e}^o \frac{d}{dt} \left[ \mathbf{R}_3(\psi) \right]_{e^o} \left[ \mathbf{R}_3(\psi) \right]_{e^o}^T \mathbf{e}^{oT}) = \]

\[ ax(\mathbf{e}^o \dot{\psi} \begin{bmatrix} -\sin \psi & -\cos \psi & 0 \\ \cos \psi & -\sin \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}^{oT}) = \]

\[ = ax(\mathbf{e}^o \dot{\psi} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{e}^{oT}) = \mathbf{e}^o \dot{\psi} \]

By use of Lemma 4.1
Euler angles, second rotation

\[ \mathbf{g} = R_1(\theta) \mathbf{f} = \mathbf{f} \left[ R_1(\theta) \right] \mathbf{f} , \]

\[ \left[ R_1(\theta) \right] \mathbf{f} = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \cos \theta & -\sin \theta \\
  0 & \sin \theta & \cos \theta 
\end{pmatrix} \]

\[ \mathbf{R} = R_3(\phi) R_1(\theta) R_3(\psi) \]

\[ \Omega_{gf} = ax \left( \frac{d}{dt} \left[ R_1(\theta) \right] \mathbf{f} \left[ R_1(\theta) \right]^T \mathbf{f}^T \right) = \]

\[ = ax \left( \mathbf{f} \dot{\theta} \begin{pmatrix}
  0 & 0 & 0 \\
  0 & -\sin \theta & -\cos \theta \\
  0 & \cos \theta & -\sin \theta 
\end{pmatrix} \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \cos \theta & \sin \theta \\
  0 & -\sin \theta & \cos \theta 
\end{pmatrix} \mathbf{f}^T \right) = \]

\[ = ax \left( \mathbf{f} \dot{\theta} \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & -1 \\
  0 & 1 & 0 
\end{pmatrix} \mathbf{f}^T \right) = \mathbf{f}_1 \dot{\theta} \]
Euler angles, third rotation

\[ \underline{e} = R_3(\phi)\underline{g} = g \left[ R_3(\phi) \right]_g, \]

\[ \left[ R_3(\phi) \right]_g = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ R = R_3(\phi)R_1(\theta)R_3(\psi) \]

\[ \omega_{eg} = ax(g \frac{d}{dt} \left[ R_3(\phi) \right]_g \left[ R_3(\phi) \right]_g^T \underline{g}^T) = \]

\[ = ax(g \dot{\phi} \begin{pmatrix} -\sin \phi & -\cos \phi & 0 \\ \cos \phi & -\sin \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \underline{g}^T) = \]

\[ = ax(g \dot{\phi} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{g}^T) = g_3 \dot{\phi} \]

127
Figure 4.10  The rolling wheel.

\[ R = R_3(\phi)R_1(\theta)R_3(\psi) \]

then by adding the relative angular velocities

\[ \omega = \omega_{ee^0} = \omega_{eg} + \omega_{gf} + \omega_{fe^0} = g_3 \dot{\phi} + f_1 \dot{\theta} + e_3^0 \dot{\psi} \]

Note that \( g_3 = e_3 \).
Exercise 1:12

Given a rotation tensor $\mathbf{R} = \mathbf{R}(\psi, \theta, \phi)$ where $\psi, \theta, \phi$ are Euler angles. Calculate the matrices $\left[ \mathbf{R}(15^\circ, 35^\circ, 60^\circ) \right]_{e_i^o}$ and $\left[ \mathbf{R}(15^\circ, 35^\circ, 60^\circ)e_i^o \right]_{e_i^o}$. 
Exercise 1:12

The rotation matrix expressed in Euler angles:

\[
R(\psi, \theta, \phi) := \begin{pmatrix}
\cos(\psi) & -\sin(\psi) & 0 \\
\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos(\theta) & -\sin(\theta) \\
0 & \sin(\theta) & \cos(\theta)
\end{pmatrix}
\begin{pmatrix}
\cos(\phi) & -\sin(\phi) & 0 \\
\sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Conversion from degrees to radians:

\[
R_d(\psi, \theta, \phi) := R\left(\psi \frac{\pi}{180}, \theta \frac{\pi}{180}, \phi \frac{\pi}{180}\right)
\]

Calculation of the rotation matrix:

\[
R_d(15, 35, 60) = \begin{pmatrix}
0.299 & -0.943 & 0.148 \\
0.815 & 0.171 & -0.554 \\
0.497 & 0.287 & 0.819
\end{pmatrix}
\]

Checking the rotation matrix:

\[
R_d(15, 35, 60) \cdot R_d(15, 35, 60)^T = \begin{pmatrix}1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1\end{pmatrix}
\Rightarrow R_d(15, 35, 60) = 1
\]

Calculating the rotation of the first basis vector:

\[
R_d(15, 35, 60) \cdot \begin{pmatrix}1 \\
0 \\
0\end{pmatrix} = \begin{pmatrix}0.299 \\
0.815 \\
0.497\end{pmatrix}
\]
Exercise 1:14

The mechanism to control the deployment of a spacecraft solar panel from position A to position B is to be designed. Determine the transplacement, i.e. the translation vector and rotation tensor $R$, which can achieve the required change of placement. The side facing the positive $x$-direction in position A must face the positive $z$-direction in position B. Calculate the rotation vector $n\varphi$ corresponding to the rotation. (Meriam & Kraige 7/1).
Exercise 1:14